April 10, 1996

Answers to Exam #2 for Math 152, sections 1 & 2

Here are the questions together with some answers to version **A** of the second exam. There are certainly other methods of getting the answers to some the questions. I hope these solutions will be useful.

(13) 1. Use Taylor's Theorem to explain how to compute $e^{-1/5}$ to accuracy of $\pm.001$; you should write down a sum which is the desired approximation, and be sure to specify how you know that the error is within the desired tolerance.

Answer Taylor's Theorem says that $f(x) = T_n(x) + \text{Error}$ where $T_n(x) = \sum_{j=0}^{N} \frac{f^{(j)}(0)}{j!}x^j$ and where Error $= \frac{f^{(N+1)}(z)}{(N+1)!}x^{N+1}$ with z between 0 and x. Here $f(x) = e^x$ so that $f^{(j)}(0) = 1$ and $f^{(N+1)}(z) = e^z$ and (since $x = -1/5$) Error $= \frac{e^z}{(N+1)!}(-1/5)^{(N+1)}$ with z between $-1/5$ and 0. Since the exponential function is increasing, $e^z \le e^0 = 1$ so $|\text{Error}| \le \frac{1}{(N+1)!}(1/5)^{(N+1)}$. You can omit the minus sign since we're interested in $\pm$ error. A small amount of experimentation will show that when $N = 3$ this is true (`maple` reports that $\frac{(.2)^4}{4!} \le .000067$) so that $e^{-1/5}$ is $\sum_{j=0}^{3} \frac{1}{j!}(-1/5)^j \pm .001$. It turns out that *any* $N \ge 3$ is a valid answer, but this approach *won't* succeed to control the error adequately if $N \le 2$.

(13) 2. What polynomial will approximate $\sin x$ with an error of $\pm.01$ on the interval $[-2, 2]$? Be sure to explain how you know that your answer is correct.

Answer The likely candidate is a Taylor polynomial. Taylor's Theorem says that $f(x) = T_n(x) + \text{Error}$ where $T_n(x) = \sum_{j=0}^{N} \frac{f^{(j)}(0)}{j!}x^j$ and Error $= \frac{f^{(N+1)}(z)}{(N+1)!}x^{N+1}$ with z between 0 and x. All the derivatives of sin are $\pm\sin$ or $\pm\cos$ and are therefore bounded in absolute value by 1 for *any* value of z. The largest x^{N+1} can be in absolute value for x in the interval $[-2, 2]$ is 2^{N+1}, so that $|\text{Error}|$ is bounded by $\frac{2^{N+1}}{(N+1)!}$. Some experimentation shows that if $N = 7$ then $\frac{2^{N+1}}{(N+1)!} \approx .0064$. Thus $T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ is a polynomial fitting the requirements of the problem. Note that *any* $N \ge 7$ is suitable.

(16) 3. Explain why the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+5}$$

converges. Find N so that the partial sum $s_N = \sum_{n=1}^{N} \frac{1}{n^2+5}$ is within 10^{-5} of the sum of the whole infinite series. Be sure to tell why the value of N you give is a correct answer.

Answer All the series in this problem are series with positive terms. Therefore if a *larger* series converges, then the *smaller* series also converges. In this case, the larger series is the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a p–series with $p = 2 > 1$ so that the larger series converges. The error estimate desired needs more details.
The difference between s_N and the sum of the whole series (the "infinite tail") is overestimated by $\sum_{n=N+1}^{\infty} \frac{1}{n^2}$ which is itself overestimated by $\int_N^{\infty} \frac{1}{x^2}\,dx = \frac{1}{N}$ (here I am considerably abridging the computation of an

improper integral!). This reasoning says that the infinite tail will be less than 10^{-5} if N is at least 10^5. So if $N \geq 10^5$ the partial sum satisfies the requirements of the problem.

(16) 4. a) Write a power series which is equal to $\ln(1+x)$ for x near 0.

Answer $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$

b) Write a power series which is equal to $\ln(1+x^3)$ for x near 0.

Answer $\ln(1+x^3) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{3n}}{n}$

c) Write $\int_0^{1/4} \ln(1+x^3)\,dx$ as a series using your answer to b).

Answer Integrate each term (a simple monomial) and "plug in" the limits of integration (note that the lower limit, $x=0$, contributes nothing). The integral then becomes $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1/4)^{3n+1}}{n(3n+1)}$

(12) 5. Determine whether any of the following series converge, and find their sums if they do.
Give an **exact** *answer, which may be an expression like "$\sqrt{7}\ln 5+\pi$." A* **numerical** *approximation like "4.25817" is* **not acceptable**.

a) $\sum_{n=1}^{\infty}\frac{4^n}{3^{2n}}$

Answer This is a geometric series with first term 4/9 and ratio between successive terms also 4/9 which is, of course, less than 1 so that the series converges. We have a formula for the sum of a convergent geometric series. The sum must be $\frac{4/9}{1-(4/9)}$ which is $\frac{4}{5}$.

b) $\sum_{n=0}^{\infty}\frac{4^n}{n!}$

Answer The Ratio Test can be applied to this series to show that it converges (this is easy). Even easier is recognizing that this is the Taylor series for e^x when $x=4$. This series converges, and its value is e^4.

(20) 6. For each of the following power series, determine the values of x for which the series converges, and the values of x for which the series diverges. Tell what tests you are using, and how the tests justify your answers, and whether the convergence is absolute or conditional.

a) $\sum_{n=0}^{\infty}\frac{2^n x^n}{(n!)^2}$

Answer Use the Ratio Test. Here $a_n = \frac{2^n x^n}{(n!)^2}$ so $a_{n+1} = \frac{2^{n+1}x^{n+1}}{((n+1)!)^2}$. We consider the absolute value of the ratio:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}x^{n+1}\,(n!)^2}{2^n x^n\,((n+1)!)^2}\right| = \left|\frac{2x}{(n+1)^2}\right|$$

The fact that $(n+1)! = (n+1)n!$ is used as usual. The limit of the ratios as $n\to\infty$ is **0** for *every* x. Therefore the series converges absolutely and converges for every x.

b) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(\frac{x}{5}\right)^n$

Answer Again let's begin with the Ratio Test. If $a_n = \frac{1}{\sqrt{n}}\left(\frac{x}{5}\right)^n$ then $a_{n+1} = \frac{1}{\sqrt{n+1}}\left(\frac{x}{5}\right)^{n+1}$. The absolute value of the ratio is:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n}}{\sqrt{n+1}}\left(\frac{x}{5}\right)\right|$$

Some of the intermediate algebra steps have been skipped. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n}}{\sqrt{n+1}}\left(\frac{x}{5}\right)\right| = \sqrt{\lim_{n\to\infty}\left(\frac{n}{n+1}\right)}\left|\frac{x}{5}\right| = \left|\frac{x}{5}\right|$$

From this we can conclude:
Divergence If $\left|\frac{x}{5}\right| > 1$ then the series diverges. So if $x > 5$ or if $x < -5$ the series diverges.

Convergence If $\left|\frac{x}{5}\right| > 1$ then the series converges absolutely and converges. So if $-5 < x < 5$ the series converges absolutely and converges.
The Ratio Test does not give enough information if $x = \pm 5$, and these cases must be investigated separately.
If $x = 5$ the series becomes $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which diverges (this is a $p-$series with $p = \frac{1}{2} < 1$).
If $x = -5$ the series becomes $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(-1)^n$. This series satisfies all the hypotheses of the Alternating Series Test (the signs of the terms alternate, the terms decrease in absolute value, and their limit is 0). So the series converges, but the convergence is conditional, since if the signs are dropped we get the series when $x = 5$ which diverges.

(10) 7. True or false? If false, give an example to show that the implication is not true.

a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer This is false. The harmonic series, $\sum_{n=1}^{\infty}\frac{1}{n}$, is an example of a series which diverges but which has n^{th} term tending to 0.

b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Answer This is true.