February 20, 1996

Review Problems for Exam #1 in Math 152:09-11

1. Find the following indefinite integrals (you can check most of your answers using the integral formulas on the back flyleaf of the text).

a) $\displaystyle\int x^2 \sin x \, dx$ b) $\displaystyle\int x^2 \arcsin x \, dx$

c) $\displaystyle\int x^2 (\ln x)^2 \, dx$ d) $\displaystyle\int \frac{x^2 - x + 3}{(x+1)(x^2+1)} \, dx$

2. a) Sketch the region R in the first quadrant enclosed by the curves $y = x^4$ and $x = y^2$.

b) Compute the area of the region R.

c) Compute the volume which results when the region R is revolved about the $x-$axis.

d) The base of a solid is the region R. Cross-sections of the solid perpendicular to the $y-$axis are squares. What is the volume of the solid?

e) The region R is divided into two parts by the straight line $y = mx$ when $m > 0$. Find m so that each part has the same area.

3. Suppose $f(x)$ is a function with the following properties:

i) $f(0) = 2$.

ii) The average value of the function $f(x)$ over any interval $[a, b]$ is $3\left(\dfrac{f(b) - f(a)}{b - a}\right)$.

Find a formula for $f(x)$. (*Hint:* Write condition ii) in terms of an integral, and then take the derivative $\dfrac{d}{db}$ of both sides.)

4. a) Use an appropriate trig substitution to express $\displaystyle\int \frac{1}{\sqrt{x^2 - 4}} \, dx$ in terms of 'elementary' functions of x (the functions on your calculator).

b) Use your result from a) to compute $\displaystyle\int_5^6 \frac{1}{\sqrt{x^2 - 4}} \, dx$. First express the answer 'exactly' in terms of values of 'elementary' functions (such as $\arctan(\pi + \sqrt{5})$); then obtain a decimal answer with your calculator.

c) Use Simpson's rule to approximate $\displaystyle\int_5^6 \frac{1}{\sqrt{x^2 - 4}} \, dx$. Find (by experimentation) the number of subdivisions n so that your answer agrees with the answer in b) to 9 decimal places (the maximum accuracy of your calculator).

5. Let $I = \displaystyle\int_0^1 \cos\left(e^{-x}\right) \, dx$

a) Use the trapezoidal rule with $n = 5$ subdivisions to find an approximate numerical value of the integral.

b) Let T_n be the approximation to I given by the Trapezoidal Rule with n subdivisions. Use the error estimate for the Trapezoidal rule to find a value of n such that $|I - T_n| < 10^{-5}$.

6. Determine whether each of the following is convergent or divergent, and evaluate those that are convergent. Be sure to show your work and explain your reasoning.

a) $\displaystyle\int_1^e \frac{1}{x\,(\ln x)^2}\,dx$ b) $\displaystyle\int_e^\infty \frac{1}{x\,(\ln x)^2}\,dx$

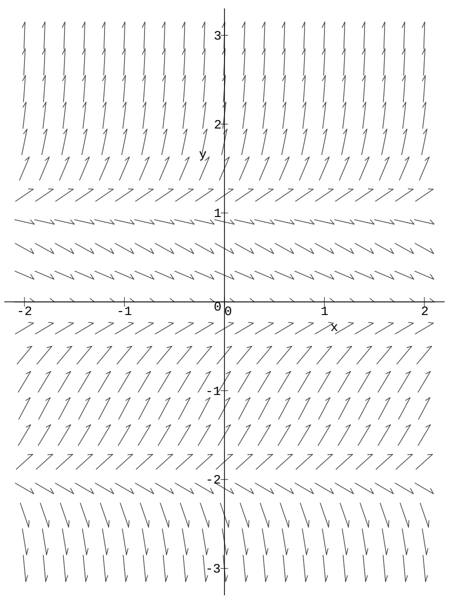
c) $\displaystyle\int_2^4 \frac{1}{\sqrt{x-2}}\,dx$ d) $\displaystyle\int_0^{\pi/4} \frac{1}{(x^2+1)\,\tan x}\,dx$

7. a) Find the equation for the curve that passes through the point $x = 0$, $y = 2$ and whose slope at any point (x, y) is $y(y + 1)$.

b) Plot the graph of the curve in a) in the range $0 \le x \le 2$. Find any vertical asymptotes.

8. Find all solutions of the differential equation $e^{-y}\dfrac{dy}{dx} + \cos x = 0$.

9. Consider the differential equation $y' = y(y + 2)(y - 1)$. A "sketch" by `maple` of its direction field is below:



Consider solutions of this differential equation subject to the initial condition

$$y(0) = y_0$$

For different values of y_0 consider $\lim_{x\to+\infty} y(x)$. Describe the asymptotic behavior you observe as a function of y_0. Be as precise as you can. Sketch appropriate examples of solution curves exhibiting each kind of asymptotic behavior.

You should begin by discussing for which values of K the "curves" (horizontal lines!)

$$y = K$$

are solutions of this differential equation.

Note This is a set of review problems, not a real exam! There are more questions asked here than there will be on the exam. Moreover, although these problems are generally similar to exam problems, it is possible that the exam will contain some problems quite different from any here. You will be allowed to bring to and use on the exam one 8 1/2 × 11 sheet of formulas and a calculator.

Math 152 Review Problems for Exam #2

1. Determine whether $\lim_{n\to\infty} a_n$ exists when a_n is defined as follows. Find the limit if it exists. Show your reasoning.

a) $a_n = \dfrac{\ln(2+3e^n)}{5n}$ b) $a_n = \dfrac{(n+2)!}{n!(3+5n)^2}$ c) $a_n = \sqrt{3a_{n-1}+4}$ and $a_1 = 2$

2. Determine whether any of the following series converge, and find their sums when they exist. Show your reasoning.

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$ b) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+4^n}{5^n}$ c) $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{(-1)^n}{n^2}\right)$

3. Do the following series converge? Do they converge absolutely? Do they converge conditionally? In each case that the series converges, find a value of N so that the sum of the terms for $N < n < \infty$ is less than 10^{-3}. Show your reasoning.

a) $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\sqrt{n}\ln n}$ b) $\displaystyle\sum_{n=1}^{\infty} n^3 e^{-n^4}$ c) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\left(\frac{1+n^2}{1+3n^2}\right)$

4. Use the ratio test or the root test to find the radius of convergence of each of the following power series. Also test the endpoints to find the complete interval of convergence for the series. Show your reasoning.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^3x^n}{3^n}$ b) $\displaystyle\sum_{n=1}^{\infty}\frac{5^nx^n}{n^{1/4}}$ c) $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{(3n+1)!}x^n$

5. Write the value of $\displaystyle\int_0^1 e^{-x^2}\,dx$ as a series. Then calculate a partial sum of the series which is within 10^{-8} of the value of the definite integral. Show your reasoning.

6. Find

$$\lim_{x\to 0}\frac{\left(e^{3x}-1\right)^4}{\cos(x^2)-1}$$

and explain the process used. (*Hint:* Use Taylor series instead of l'Hospital's rule.)

7. Find the Taylor series centered at 0 (the Maclaurin series) for the following functions:

a) $\dfrac{1}{1+x^3}$ b) $x\sin(2x)$ c) $\dfrac{\ln(1+x^2)}{x}$

8. Find the terms up to and including the x^4 term of the Taylor series centered at 0 (the Maclaurin series) for the following functions:

a) $\dfrac{\sin x}{1+x}$ b) $\dfrac{e^x-e^{-x}}{2}$ c) $\left(1+x^{10}\right)^{1/7}$ d) $e^{(x^2)}$

9. Find a polynomial $P(x)$ so that the difference

$$|P(x) - \cos x|$$

will be *less than* 10^{-3} for all x in the interval $[-2, 2]$.

10. Suppose that f is a function whose fourth derivative satisfies the inequality

$$\left| f^{(4)}(x) \right| \leq \frac{1}{3 + 5x^2}$$

for all x and that $T_3(x)$ is the third degree Taylor polynomial for f centered at 0. Find an estimate for the error obtained when using $T_3(2)$ in place of $f(2)$.

Spring, 1997

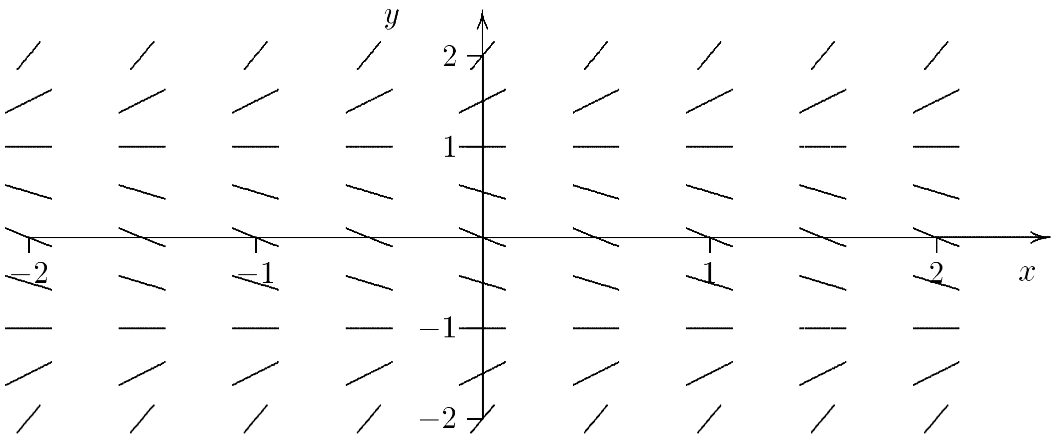
Math 152 & 154 final exam review problems

The final exam will be cumulative. Please be sure to include in your review for the final exam the review problems for the midterm exams, the midterm exams, and the workshop problems. Note also that the problems below are intended *only* to be representative of the general type of problem which may be asked on the final, and are in no particular order.

1. Solve the following differential equations with initial conditions:

a) $x\dfrac{dy}{dx} = -y$, with $y(4) = 2$ b) $\dfrac{dy}{dx} = 1 + y^2$ with $y\left(\dfrac{\pi}{3}\right) = 1$

2. Here is a sketch of the direction field for the differential equation $\dfrac{dy}{dx} = y^2 - 1$:



Answer the following questions *without* solving the differential equation.

a) On the graph above sketch a solution of the equation that satisfies the condition $-1 < y(0) < 1$. Find $\lim\limits_{x\to\infty} y(x)$ and $\lim\limits_{x\to-\infty} y(x)$ for such a solution.

b) On the graph above sketch a solution of the equation that satisfies the condition $y(0) < -1$. Find $\lim_{x\to\infty} y(x)$ and $\lim_{x\to-\infty} y(x)$ for such a solution.

3. Does the improper integral $\displaystyle\int_0^\infty \frac{x^2}{1+x^3}\,dx$ converge or diverge?

4. For each of the following series, find all values of $p > 0$ for which the series is convergent (the answer might be different in each case):

a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^p}$ b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ c) $\displaystyle\sum_{n=1}^{\infty} \left[1 - \cos\left(\frac{1}{n}\right)\right]^p$

(*Hints:* a) alternating, b) integral, c) Taylor series for $\cos(x)$ and compare.)

5. a) Use the formula for the sum of a geometric series to obtain the power series centered at 0 for the function $\frac{x}{5+x^3}$. What is the radius of convergence of this series?

b) Use your answer to b) to write $\int_0^x \frac{x}{5+x^3}\, dx$ as an infinite series.

c) Find a specific partial sum of this series which is equal to the definite integral with an error of at most 10^{-8}.

6. Show that $\sum_{n=0}^{\infty} \frac{3}{\sqrt{n^5+7}}$ converges and that $\sum_{n=0}^{100} \frac{1}{\sqrt{n^5+7}}$ approximates the sum of the full infinite series in with an error of less than 10^{-3}. (*Hint:* Compare what's *not* in the partial sum to an integral.)

7. Find a positive number A so that if x is in the interval $[0, A]$ then

$$\left|\ln(1+x) - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)\right| < \frac{1}{100}$$

Be sure to explain how you know that your answer is correct.

8. Let $f(x) = \sqrt{1+x}$

a) Find the Taylor polynomial $T_2(x)$ of degree 2 (centered at $a = 0$) for $f(x)$.

b) Find a number M so that $|f^{(3)}(z)| \leq M$ for all z in the interval $[-1/2, 1/2]$. Justify your choice of M (an answer such as $10^{10^{10}}$ is not acceptable).

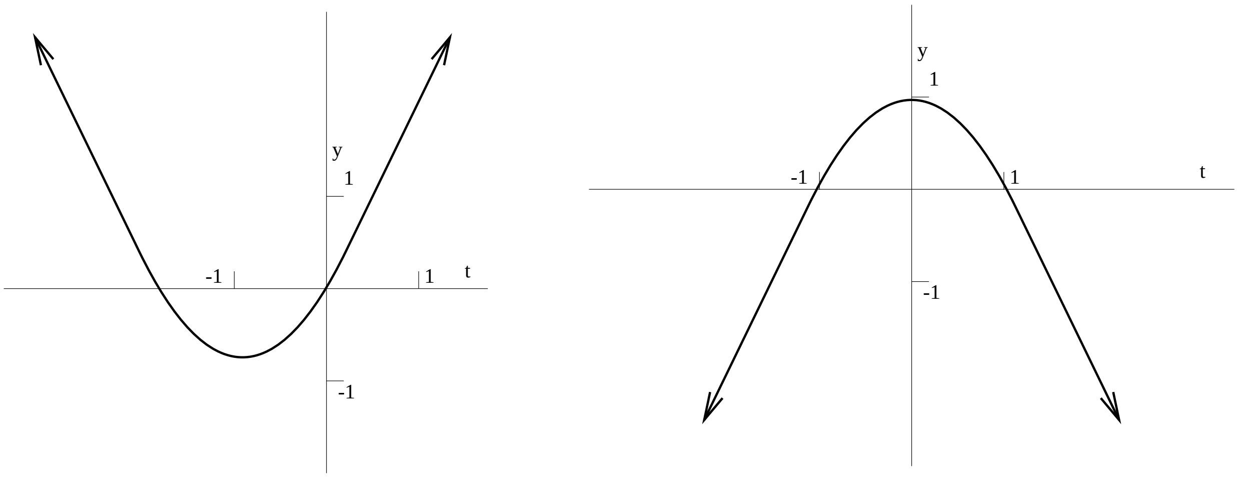
c) Use the remainder formula in Taylor's theorem and the value of M that you found in b) to find a number $B \leq 1/2$ so that $|f(x) - T_2(x)| < 10^{-8}$ for all x in the range $-B \leq x \leq B$.

9. a) Sketch the polar curves $r = 1 + \theta$ and $r = 3\sin(2\theta)$ for $0 \leq \theta \leq \frac{\pi}{2}$.

b) The curves intersect twice. Find the approximate values of θ at which they intersect.

c) What is the area of the region touching the origin and bounded by the two curves?

10. A curve is described by parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ where the graphs of x as a function of t and y as a function of t are given below.



x=f(t) y=g(t)

a) What is the (approximate) position of the point on the curve when $t = \dfrac{1}{2}$?

b) What is the (approximate) slope of the line tangent to the curve when $t = \dfrac{1}{2}$?

c) Find an equation for the line tangent to this curve when $t = \dfrac{1}{2}$ using the information you have obtained in a) and b).

d) Sketch the curve as well as you can.

11. Use a preliminary substitution or a trig identity, followed by integration by parts, to compute the following definite integrals:

a) $\displaystyle\int_0^4 e^{\sqrt{x}}\,dx$ b) $\displaystyle\int_0^{\pi/2} x\sin(x)\cos(x)\,dx$ c) $\displaystyle\int_0^1 x^3\arctan(x^2)\,dx$

12. a) Sketch the region in the first quadrant which is bounded by the curves $y = x^2$ and $x = y^6$.

b) Find the area of the region sketched in a).

c) Find the volume of the solid which results when the region sketched in a) is revolved around the $x-$axis.

13. Use partial fractions to prove that $\displaystyle\int_1^2 \frac{x^3+x+1}{x^2+x}\,dx = \ln 3 + \frac{1}{2}$.

14. Find the following indefinite integrals:

a) $\displaystyle\int \frac{dx}{(9+x^2)^{3/2}}\,dx$ b) $\displaystyle\int (\sec x)^2 \tan x\,dx$ c) $\displaystyle\int \frac{e^x}{1+e^x}\,dx$