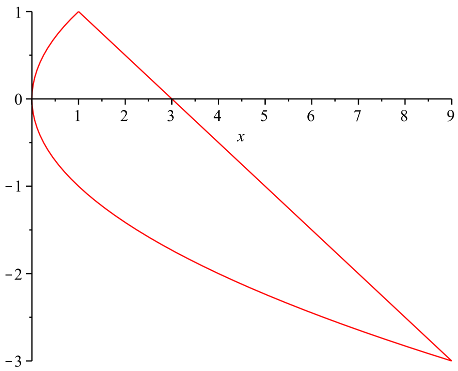
Solutions to Midterm I, yellow version of the test.
Problems 1-5 worth 14 points each, problems 6-8 worth 10 points each.
Median scores: #1. 12; #2. 7; #3. 8; #4. 11; #5. 11; #6. 0; #7. 2; #8. 6.
Median test grade: 54 (half the students did better, half did worse).

---

1.[14] $(a)$ Sketch the region between the parabola $x = y^2$ and the line $x + 2y = 3$, and $(b)$ compute its area.

$(a)$ (Sketch)



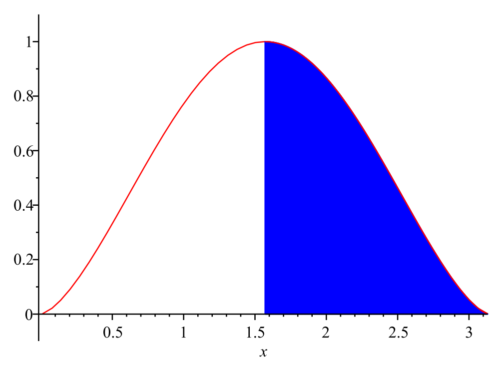

$(b)$ (Area)

Integrating along the $y$-axis: $\displaystyle\int_{-3}^{1} (3-2y) - y^2\,dy = 3y - y^2 - y^3/3|_{-3}^{1} = 10\frac{2}{3}$.

Integrating along the $x$-axis: $\displaystyle\int_0^1 (\sqrt{x} - (-\sqrt{x}))\,dx + \int_1^9 (3-x)/2 - (-\sqrt{x})\,dx$ (harder, but reasonable).

2.[14] The region $\mathcal{R}$ lies under the curve $y = 2\sin^{3/2}(x)$ and above the interval $[\pi/2, \pi]$ on the $x$-axis.

$(a)$ Shade that region in the diagram shown on the right:



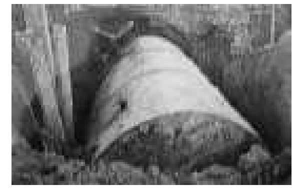

$(b)$ If the region $\mathcal{R}$ is rotated around the $x$-axis, find the volume of the resulting solid.

$$V = \int_{\pi/2}^{\pi} \pi(2\sin^{3/2}(x))^2\,dx = 4\pi\int_{\pi/2}^{\pi} \sin^3(x)\,dx$$

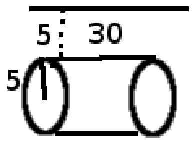
$u = \cos(x),\ du = -\sin(x)\,dx.$

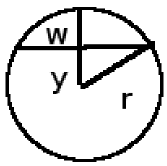
$$V = 4\pi\int_0^{-1} (1-u^2)(-1)\,du = 4\pi(-u + u^3/3)|0^{-1} = (8/3)\pi$$

3.[14] A cylindrical gasoline storage tank ten feet in diameter and thirty feet long is buried with its axis horizontal, and the top of the tank five feet underground. Write down an integral which gives the work done against gravity in pumping out a full tank of gasoline up to ground level, if gasoline weighs 47 pounds per cubic foot.



Show clearly how you obtain your answer.




Take horizontal slices: they are rectangles, with area $A(y)$ depending on the depth $y$. Let us measure $y$ from the center of the circular base of the cylinder, so $y$ goes from $-5$ to 5 (while the depth below ground goes from $-15$ to $-5$.

We want $\int_{-5}^{5} \text{Distance} \cdot (\text{density} \cdot \text{area} \cdot\, dy)$

The area $A(y)$ of the horizontal rectangular slice is $30w = 30(2\sqrt{5^2 - r^2}$. Since ground level corresponds to $y = 10$, the distance the slice is lifted is $10 - y$. So we can fill in everything:

$$\int_{-5}^{5} (10 - y) \cdot (47 \cdot 2\sqrt{25 - y^2}(30))\, dy$$

4.[14] Compute the average value of $f(x) = (\ln(x))^2$ over the interval $[1, e^2]$, and find a point in the interval $[1, e^2]$ at which the value of the function is equal to its average value.

$(a)$ Average value $\bar{f}$: $\dfrac{1}{e^2 - 1} \displaystyle\int_1^{e^2} (\ln(x))^2\, dx$.

$u = \ln^2(x)$, $v = x$:

$\displaystyle\int \ln^2(x)\, dx = x\ln^2(x) - \int x \left(\frac{2\ln x}{x}\right)\, dx = x\ln^2(x) - 2\int \ln x\, dx = x\ln^2 x - 2[x\ln x - x]$.

(By parts again.)

$\dfrac{1}{e^2 - 1}(e^2 \ln^2 e^2 - 2e^2 \ln e^2 + 2e^2 - (0 - 0 + 2)) = \dfrac{4e^2 - 4e^2 + 2e^2 - 2}{e^2 - 1} = 2$

$(b)$ Point where $f(x) = \bar{f}$:

$\ln^2(x) = 2$, $x = e^{\sqrt{2}}$

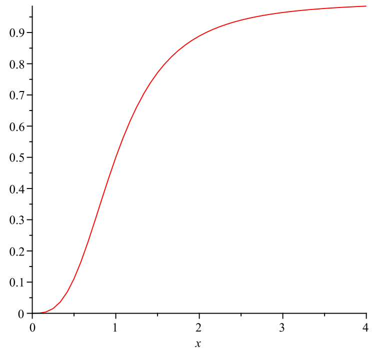
5.[14] Let $f(x) = \dfrac{x^3}{1+x^3}$.

$(a)$ Write down, but do not compute, the approximation to $\int_1^3 f(x)\,dx$ given by Simpson's rule with to a division of the interval $[1,3]$ into 6 equal subintervals.

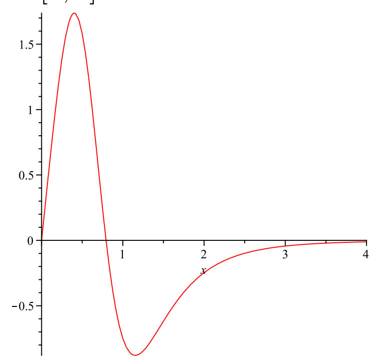
$\Delta x = 1/3$.

$(1/3)(\frac{1}{2} + 4\frac{(4/3)^3}{1+(4/3)^3} + 2\frac{(5/3)^3}{1+(5/3)^3} + 42^3/1+2^3 + 2\frac{(7/3)^3}{1+(7/3)^3} + 4\frac{(83/)^3}{1+(8/3)^3} + 1\frac{3^3}{1+3^3})(1/3)$

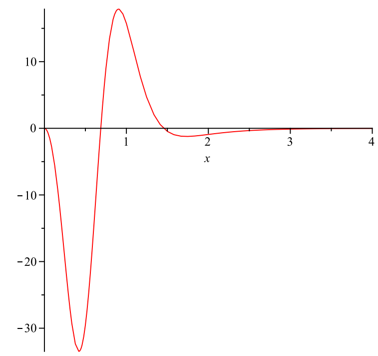
$(b)$ Estimate the error involved in the approximation in part $(a)$. The graphs of $f(x)$ and its 2nd and 4th derivatives over the interval $[0,4]$ are shown below.



f(x)



f"(x)



4th derivative

Looking at the third graph on the interval [1,3], $K_4 = 20$ will be safe.

So Error $\leq \frac{K_4(b-a)^5}{180N^4} = \frac{20\cdot 2^5}{180\cdot 6^4}$

$(c)$ If the sum in part $(a)$ is $1.681271992\ldots$, and we want to know whether the exact value is at least 1.67, how could the result of $(b)$ be helpful?

If the bound in $(b)$ is less than .01 then since the value is at least 1.671, it is certainly above 1.67.

*That's enough of an explanation, but some people pointed out that we can see that the bound in* $(b)$ *is indeed less than* .01 *without computing it:*

$$\frac{20\cdot 2^5}{180\cdot 6^4} = \frac{40}{180\cdot 3^4} < 1/180 < .01$$

6.[10] Verify that $\displaystyle\int_0^1 \frac{dx}{\sqrt{x^2+2x+5}} = \ln\left(\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right)$

$u = x+1,\ du = dx$: $\displaystyle\int_1^2 \frac{dx}{\sqrt{u^2+4}\,du}$

$u = 2\tan\theta,\ du = \sec^2\theta\,d\theta$:

$$\int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \ln\left(\frac{u}{2} + \sqrt{1+\frac{u^2}{4}}\right).$$

So $\ln\left(\frac{u}{2} + \sqrt{1+\frac{u^2}{4}}\right)\Big|_1^2 = \ln\left(\frac{1+\sqrt{2}}{1/2+\sqrt{5/4}}\right) = \ln\left(\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right)$

7.[10] Verify that $\displaystyle\int_0^{\ln 2} \frac{dx}{3-e^x} = \frac{2}{3}\ln(2)$.
*Suggestion:* Begin with the substitution $u = e^x$.
$du = e^x\,dx,\ dx = du/u$
$$\int_1^2 \frac{(du/u)}{3-u} = \int_1^2 \frac{du}{u(3-u)}.$$
$\frac{1}{u(3-u)} = \frac{A}{u} + \frac{B}{3-u}$;
$1 = A(3-u) + Bu$; $u = 0$—$A = 1/3$; $u = 3$—$B = 1/3$
$\displaystyle\int_1^2 \frac{1/3}{u} + \frac{1/3}{3-u}\,du = (1/3)(\ln u - \ln|3-u|)|1^2 = (1/3)\ln(\frac{u}{3-u})|1^2 = (1/3)(\ln 2 - \ln(1/2)) = (2/3)\ln(2)$.

8.[10] Verify that $\displaystyle\int_0^{\pi^2/4} \frac{\sin^3(\sqrt{x})\cos^5(\sqrt{x})}{\sqrt{x}}\,dx = 1/12$.
*Suggestion:* Begin with the substitution $u = \sqrt{x}$.
$du = (1/2)\dfrac{dx}{\sqrt{x}}$
$$\int_0^{\pi/2} \sin^3(u)\cos^5(u)(2\,du) = 2\int_0^{\pi/2} \sin^3(u)\cos^5(u)\,du.$$
$v = \cos u,\ dv = \sin u\,du,$
$$2\int_1^0 (1-v^2)v^5(-\,dv) = 2\int_0^1 (v^5 - v^7)\,dv = 2(v^6/6 - v^8/8)\_0^1 = 2(1/6 - 1/8) = 1/12$$