Solutions to Midterm II, pink version of the test.
Problems 1,2,4,5,7 worth 12 points each; problem 3 worth 10 points; problems 6,8 worth 15.
Median scores: #1. 5; #2. 5; #3. 7; #4. 5.5; #5. 2; #6. 8.5; #7. 9.5; #8. 6.5.
Median test grade: 49 (half the students did better, half did worse).

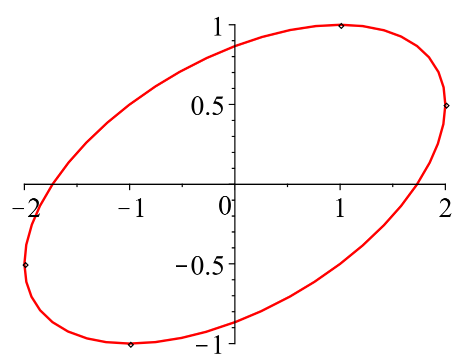
1. [12] For each of the following integrals: if it does not exist, explain why; and if it does exist, compute it.

(*a*) $\displaystyle\int_0^1 \frac{dx}{3x+x^2}$
Near 0, the denominator is approximately $3x$, so by the p-rule with $p = 1$, the integral diverges.
(*b*) $\displaystyle\int_1^\infty \frac{dx}{2x+x^2}$
By comparison with $1/x^2$ and the p-rule ($p = 2$), the series converges so we must compute it:
$$\frac{1}{2x+x^2} = \frac{1/2}{x} - \frac{1/2}{2+x}$$
$$\int_1^\infty \left(\frac{1/2}{x} - \frac{1/2}{2+x}\right) dx = (1/2)\ln\left(\frac{x}{2+x}\right)|_1^\infty = (1/2)\ln(1) - (1/2)\ln(1/3) = 1/2\ln(3)$$
(*c*) $\displaystyle\int_{-1}^1 \frac{2\cos(3x)}{x^2}\, dx.$
Near 0 this is approximately $2/x^2$ so diverges, by the p-rule.

2. [12] Consider the parametric curve defined by

$$x = \sqrt{3}\cos(t) + \sin(t)$$
$$y = \sin(t)$$

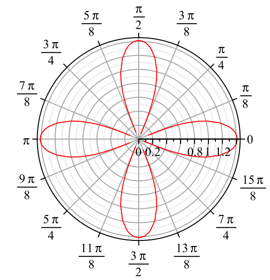
(*a*) Find the points on the curve where the tangent line is horizontal, and those where it is vertical.
$$\frac{dy}{dx} = \frac{\cos t}{-\sqrt{3}\sin(t) + \cos(t)}$$
Horizontal tangent: $\cos(t) = 0$, $\sin(t) = \pm 1$, at the points $(\pm 1, \pm 1)$.
Vertical tangent: $-\sqrt{3}\sin(t) + \cos(t) = 0$, $\tan(t) = 1/\sqrt{3}$, $\sin(t) = \pm 1/2$, $\cos(t) = \pm\sqrt{3}$, at the points $(\pm 2, \pm 1/2)$
(*b*) Sketch the curve, labelling these points clearly.



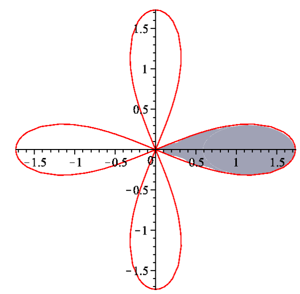
3. [10] Find the area inside one loop of the graph of the polar equation

$$r^2 = 3\cos(4\theta)$$

shown on the right (polar grid added)



(*a*) In the picture, shade the region whose area you are computing.



(*b*) Write down the appropriate integral, and explain how you chose the limits of integration

$$\int_{-\pi/8}^{\pi/8} (1/2)3\cos(4\theta)\, d\theta$$

Explanation: The loop starts at the origin and returns there as 4θ goes from $-\pi/2$ to $\pi/2$. We did not have the grid to look at.

(*c*) Calculate the integral.

$$3/2 \frac{\sin(4\theta)}{4}\bigg|_{-\pi/8}^{\pi/8} = (3/8)(2) = 3/4$$

4. [12] Show that the length of the curve given by $y = \dfrac{e^x + e^{-x}}{2}$ over the interval $[-\ln 2, \ln 3]$ is $25/12$.

$$ds/dx = \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}^2\right)} = \sqrt{1 + \frac{e^{2x} - 2 + e^{-2x}}{4}} = \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}} = \frac{e^x + e^{-x}}{2}.$$

Length: $\displaystyle\int_{-\ln(2)}^{\ln(3)} (1/2)(e^x + e^{-x})\, dx = (1/2)(e^x - e^{-x})|_{-\ln(2)}^{\ln(3)} = (1/2)(3 - (1/3)) - (1/2)(1/2 - 2) =$ $4/3 + 3/4 = 25/12$

5. [12] An 8 lb. turkey is removed from the freezer compartment at 0°F, and placed to thaw in the refrigerator at 40°F. After a day, the temperature at the center of the turkey is 32°F. Assume Newton's law of cooling (or warming) applies.

(*a*) What is the differential equation governing the temperature T of the turkey in degrees Fahrenheit, in terms of the elapsed time t (in hours)? Give any constants involved explicitly.
Differential Equation: $T' = k(T - 40)$. To determine k we start with the solution $T - 40 = Ce^{kt}$ and take $t = 0$ to find $C = -40$, then $t = 1$ to find $-8 = -40e^k$, and $k = \ln(1/5)$, so the differential equation is

$$T' = \ln(1/5)(T - 40)$$

–or $T' = -\ln(5)(T - 40)$, which makes it clearer that k is negative.

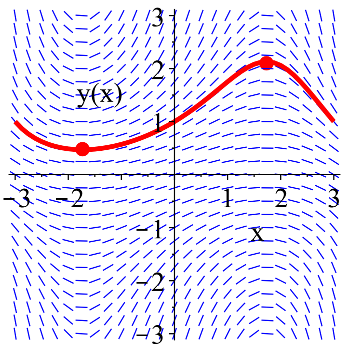
5(b) What is the exact temperature of the turkey after 36 hours?
That would mean $t = 3/2$, $T - 40 = -40e^{(3/2)\ln(1/5)}$, or

$$T = 40\left(1 - e^{(3/2)\ln(1/5)}\right)$$

which can be written in various excellent ways, but not in any very simple way.

6. [15]

(a) The slope field for the differential equation $\dfrac{dy}{dx} = \dfrac{(3-x^2)(1+y^2)}{10}$ over the range $x, y \in [-3,3]$ is shown below. Sketch the solution to the corresponding initial value problem with $y(0) = 1$ over the interval $[-3,3]$, and indicate any critical points.



(Critical points: $x = \pm\sqrt{3}$.)

(b) Find an *explicit formula* of the form $y = f(x)$ describing the curve drawn above.

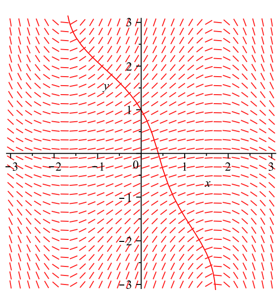
$\displaystyle\int \frac{dy}{1+y^2} = \int \frac{3-x^2}{10}\,dx$

$\tan^{-1}(y) = (1/10)(3x - \frac{x^3}{3}) + C$

$\tan^{-1}(1) = C$; $C = \pi/4$

$y = \tan(\dfrac{3x - (x^3/3)}{10} + \pi/4)$

(c) Below, sketch a curve passing through the point $(0,1)$, with tangent line *perpendicular* to the elements of the slope field at every point.



What is the differential equation satisfied by this curve?

$$y' = \frac{-10}{(3-x^2)(1+y^2)}$$

7. [12] Let $T_3(x)$ be the Taylor polynomial of degree 3 for the function $y = \sin(x)$ at $a = \pi/3$.

(a) Calculate $T_3(x)$ explicitly.
$\sin(\pi/3) + \cos(\pi/3)(x - \pi/3) - (\sin(\pi/3)/2)(x - \pi/3)^2 - (\cos(\pi/3)/6)(x - \pi/3)^3$ or
$\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}(x - \pi/3) - \dfrac{\sqrt{3}}{4}(x - \pi/3)^2 - \dfrac{1}{12}(x - \pi/3)^3$.

(b) If the value $T_3(1)$ is used as an approximation to $\sin(1)$, estimate the error involved.

As the 4th derivative of $\sin(x)$ is $-\cos(x)$, bounded by $K = 1$, the error is at most $(1)|1-\pi/3|^4/4! \approx (.047)^4/24$ which is quite small (considerably less than $.1^4/24 = .0001/24$ for example).

8. [15] For each of the following infinite series, determine whether it *diverges, converges conditionally, or converges absolutely.* Circle your answer, and state what tests are used: Geometric Series, p-Series, Integral Test, Comparison Test, Limit Comparison Test, Leibniz Test for Alternating Series.
(Make any calculations you need below.)

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{2n-1}$ *Divergent* **Conditionally Convergent** *Absolutely Convergent*

Tests Used: First, Limit comparison and p-series, $p = 1$; then, Leibniz

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{2n^2-1}$ *Divergent* *Conditionally Convergent* **Absolutely Convergent**

Tests Used: Limit comparison, p-series with $p = 2$.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{e^n+1}{\pi^n+1}$ *Divergent* *Conditionally Convergent* **Absolutely Convergent**

Tests Used: Limit comparison with geometric series, ratio $e/\pi < 1$.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{e^{-n}+1}{\pi^{-n}+1}$ **Divergent** *Conditionally Convergent* *Absolutely Convergent*

Tests Used: Really, the Divergence Test. But you can think of it as a comparison with the series $1 + 1 + 1 + 1 + \ldots$ which is a p-series for $p = 0$ and a Geometric Series with ratio 1 (too big!).

(e) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n[\ln(n)]^2}$ *Divergent* *Conditionally Convergent* **Absolutely Convergent**

Tests Used: Integral Test and p-test, $p = 2$: $\displaystyle\int_2^{\infty} \frac{dx}{x\ln(x)^2} = \int_{\ln(2)}^{\infty} \frac{du}{u^2}$

(With $u = \ln(x)$, $du = \frac{1}{x}\,dx$.)