Part I. Integration and Differential Equations

(1) Improper Integrals. Evaluate those of the following integrals which converge.

(*a*) $\displaystyle\int_1^\infty \frac{dx}{x+x^3}$

$\frac{1}{x+x^3} = \frac{1}{x} - \frac{x}{1+x^2}$: the indefinite integral is $\ln(x) - \frac{1}{2}\ln(1+x^2) = \ln\left(\frac{x}{\sqrt{1+x^2}}\right)$, and the definite integral is $\ln\left(\frac{x}{\sqrt{1+x^2}}\right)\Big|_1^\infty$.

Since $\lim_{b\to\infty} \frac{x}{\sqrt{1+x^2}} = 1$, the integral converges to $\ln(1) - \ln(1/\sqrt{2}) = \frac{1}{2}\ln 2$.

(*b*) $\displaystyle\int_0^\infty \frac{x^2 dx}{e^{2x}}$

After the substitution $u = 2x$, this is equal to $(1/8)\displaystyle\int_0^\infty \frac{x^2 dx}{e^x}$.

Integrating by parts twice, $\displaystyle\int x^2 e^{-x} dx = x^2(-e^{-x}) - \int (2x)\cdot(-e^{-x})dx = -x^2e^{-x} + 2\int xe^{-x}dx = -x^2e^{-x} + 2(x(-e^{-x}) - 2\int 1\cdot(-e^{-x})dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x}$

The limit of this function as $x \to \infty$ is 0, and at $x = 0$ it equals -2, so $(1/8)\displaystyle\int_0^\infty x^2e^{-x}dx$ is 1/4.

(*c*) $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}$

$\tan^{-1}(x)|_{-\infty}^\infty = (\pi/2) - (-\pi/2) = \pi$.

(*d*) $\displaystyle\int_0^{\pi/2} \sec x dx$

$\ln|\sec x + \tan x||_0^{\pi/2}$: as $x \to \pi/2^-$, we have $\sec x + \tan x = (1+\sin x)/\cos x$, which diverges toward $+\infty$; and so the integral diverges.

(*e*) $\int_{-1}^1 \frac{1}{x^2}dx$

As this has a discontinuity at 0, the integral must be written as $\displaystyle\int_{-1}^0 \frac{1}{x^2}dx + \int_0^1 \frac{1}{x^2}dx$. These integrals diverge by the p-Test.

(*f*) $\displaystyle\int_0^1 \ln x dx$

$\displaystyle\int_0^1 \ln x dx = (x\ln x - x)|_0^1 = (-1) - \lim_{a\to 0^+} a\ln a = -1$:
$\lim_{a\to 0^+} a\ln a = 0$ by l'Hôpital's rule.

(*g*) $\int_3^\infty \frac{dx}{(x^3-x)^{1/4}}$

By comparison with a suitable multiple of $1/x^{3/4}$, this diverges.

The comparison goes like this: $x \le x^3/9$, $(x^3-x)^{1/4} \ge ((8/9)x^3)^{1/4} = (8/9)^{1/4}x^{3/4}$, $\frac{1}{x^3-x} \le (9/8)^{1/4}\frac{1}{x^{3/4}}$.

Really, it is just a variation on the Limit Comparison Test.

(*h*) $\int_0^\infty \frac{\ln(x)}{1+x^2}dx$

With the change of variable $u = \ln x$ this becomes $\int_{-\infty}^\infty \frac{ue^u du}{1+e^{2u}} = \int_{-\infty}^\infty \frac{udu}{e^u+e^{-u}}$

This converges by comparison with ue^{-u} on $[0,\infty)$ and with ue^u on $(-\infty,0]$.

As this is an odd function integrated on a symmetrical interval, the value is 0.

(2) Arc length

Find the perimeter of the cardioid $r = 1-\cos\theta$.

$dr/d\theta = \sin\theta$, $ds = \sqrt{(1-\cos\theta)^2+\sin^2\theta} = \sqrt{2-2\cos\theta}$. As $\sin^2(\theta/2) = \frac{1-\cos\theta}{2}$, we find $2-2\cos\theta = 4\sin^2(\theta/2)$ and $ds = 2|\sin(\theta/2)|$.

So we can find the perimeter: $\int_0^{2\pi} ds = \int_0^{2\pi} 2\sin(\theta/2)d\theta = -4\cos(\theta/2)|_0^{2\pi} = 8$.

(3) Arc length

Our goal in this problem is to design a test question of the following type: *"Find the length of the graph of the function $y = cx^2 - \ln x$ over the interval $[2,3]$."*

First we must find a value of c for which this problem can be solved exactly; and then we want to know the answer.

$y' = 2cx-(1/x)$, $ds = \sqrt{1+(2cx-(1/x))^2}$.

We need $1+(2cx-(1/x))^2$ to be a perfect square. We will choose c so that

$$1+(2cx-(1/x))^2 = (2cx+(1/x))^2$$

and this boils down to $1-4c = 4c$ or $c = 1/8$.

So with $c = 1/8$, the arc length of the curve given by $y = x^2/8 - \ln x$ over $[2,3]$ is $\int_2^3 \sqrt{(2(1/8)x+1/x)^2}dx = \int_2^3 (x/4+1/x)dx = 5/8+\ln(3/2)$.

(4) Parametric equations

Find a parametrization $x = f(t)$, $y = g(t)$ of the ellipse $9x^2 + 4y^2 = 36$.

Since this is equivalent to $(x/2)^2 + (y/3)^2 = 1$, it is natural to take $x/2 = \cos t$, $y/3 = \sin t$, so

$$x = 2\cos t; \qquad y = 3\sin t$$

(5) Self-intersections and tangent lines.

(a) The curve given parametrically by $x = t^2 - 9$, $y = t^3 - t$ crosses itself at a point P. Find the coordinates of that point, and the angle between the curves at that point.

The point P: Suppose we pass through P at $t = t_1$ and $t = t_2$ (with $t_1 \neq t_2$). Then $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$.

The first equation gives $t_2 = -t_1$, and then the second gives $y(t_1) = 0$. So $t_1 = 0$, 1, or -1, and $t_2 = -t_1$. Hence $t_1, t_2 = \pm 1$, in some order, and the point P is $(-8, 0)$.

The angle between the curves

First we find the tangent lines $t = 1$ and $t = -1$. We have $dy/dx = \dfrac{3t^2 - 1}{2t}$. The slopes at $t = \pm 1$ are 1 and -1. The lines are $y = x+8$ and $y = -(x+8)$.

These lines are perpendicular, so the curves are perpendicular.

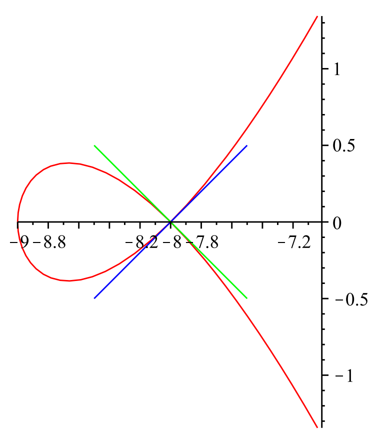

Figure 1: A Self-Intersection

(6) Area

Find the area between the circle $r = 2\cos\theta$ and the cardioid $r = 1 + \cos\theta$.

The circle is inside the cardioid, so we calculate the area of the cardioid and subtract π.

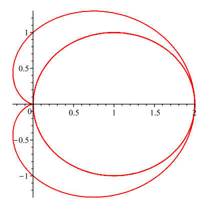

Figure 2: Circle inside a cardioid

Cardioid: $\int_0^{2\pi} (1/2)(1+\cos\theta)^2 d\theta = (3/2)\pi$ after expanding and integrating each term.

Cardioid–circle: $(3/2)\pi - \pi = \pi/2$

(7) Surface area

Find the area of the surface obtained by rotating the curve $y = \sqrt{x+1}$, $1 \le x \le 5$ about the x-axis.

$ds = \sqrt{1 + \frac{1}{4(x+1)}}dx = \sqrt{\frac{4x+5}{4(x+1)}}dx$

$\int_1^5 \left(2\pi\sqrt{x+1}\right) \cdot \sqrt{\frac{4x+5}{4(x+1)}}dx = \pi \int_1^5 \sqrt{4x+5}dx$

We make the substitution $u = 4x+5$, and the integral comes out to $(49/3)\pi$.

(8) Volume

R is the region lying within the cardioid $r = 1 + \cos\theta$, to the right of the y-axis. Write down a formula for the volume of the solid of revolution obtained by rotating the region R about the x-axis.

Use the polar equation for the curve to write the volume integral $\int_0^2 \pi y^2 dx$ in terms of θ. Since $y^2 = r^2\sin^2\theta = (1+\cos\theta)^2\sin^2\theta$ and $dx = \frac{dx}{d\theta}d\theta = -\sin\theta(1+2\cos\theta)d\theta$, the integral transforms into $\pi\int_{\pi/2}^0 (-1)\sin^3\theta(1+\cos\theta)^2(1+2\cos\theta)d\theta$.

(Note: Letting $u = 1+\cos(\theta)$, the integral becomes $\pi\int_1^2 (2-u)u^3(2u-1)du = (5/2)\pi$.)

(9) Initial value problems. In each case, find the solution explicitly.

(a) $dx/dt = \tan x$, $x(0) = \pi/6$

$\int \cot x dx = \int dt$; this leads to $\sin x = Ce^t$ as the general solution, and $x = \sin^{-1}((1/2)e^t)$ as the particular solution.

(b) $(4 + x^3)^{1/2}(dy/dx) = (xy)^2$, $y(0) = -1$

$$\int \frac{dy}{y^2} = \int \frac{x^2 dx}{(4 + x^3)^{1/2}}$$

This leads to $y = \frac{-1}{(2/3)(4+x^3)^{1/2}+C}$ for the general solution, and $C = -1/3$, $y = \frac{-3}{2\sqrt{4+x^3}-1}$ for the particular soltuion

(10) Heat transfer

If coffee served at 50°C in a room with temperature 25°C cools to 30°C in an hour, how long did it take to cool to 35°C?

The *temperature difference* $T - 25$ satisfies $T - 25 = Ce^{kt}$ for constants C and k.

Taking $t = 0$ we see $C = 25$. Taking $t = 1$ we have $5 = 25e^k$, $k = -\ln(5)$,
So when the customer drank the coffee, $10 = 25e^{-\ln(5)t}$, $t = \frac{-\ln(2/5)}{\ln(5)}$.
On the calculator this becomes .569 hours—34 minutes and 10 seconds.

(11) Celestial mechanics and mathematical finance

For reasons known only to her, an astronomer invests all her funds at 5% interest, and makes withdrawals at the rate of \$20,000 per year. If the balance will decline to 0 in 30 years, what was the initial investment?

$P' = .05P - 20,000$; $P' = .05(P - 400,000)$; $P = 400,000 + Ce^{.05t}$
$0 = 400,000 + Ce^{.05(30)}$; $C = -400,000/e^{1.5}$
Ans.: $P(0) = 400,000 + C = 400,000(1 - 1/e^{1.5})$.
(about \$310,000, says the calculator)

(12) Graphical Methods: Slope Fields

Sketch the slope field of $\dot{y} = t^2 - y$ for the region defined by $t \in [-4, 4]$, $y \in [-16, 16]$, and then sketch the graphs of the corresponding initial value problems with $y(0)$ equal to -2, 1, or 2. Discuss the critical points on these three curves.

The curves are shown below (the scales on the t- and y-axes are different). The curves do not cross, but they appear to approach one another very closely on the right. We can see one critical point on the top curve, two on the middle curve, and none on the bottom curve.

The critical points satisfy $y = t^2$. Two of them are close to $(1, 1)$, the other one is not far from $(-2, 4)$:

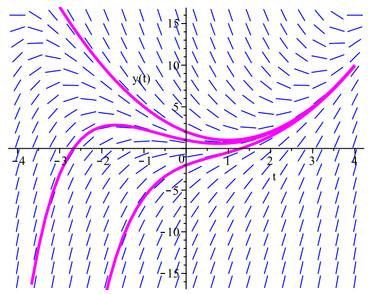

Figure 3: Slope field with integral curves

Part II. Taylor Polynomials, Sequences, and Series

(13) Taylor Polynomials and Infinite Series

(a) Find the Taylor polynomial of degree 5, $T_5(x)$, for the function $f(x) = \ln(x)$, centered at $x = 1$, and evaluate $T_5(3/2)$.

n	0	1	2	3	4	5
$f^{(n)}$	$\ln(x)$	$1/x$	$-1/x^2$	$2/x^3$	$-6/x^4$	$24/x^5$
$f^{(n)}(a)/n!$	0	1	-1/2	1/3	-1/4	1/5

$T_5 = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3 - (1/4)(x-1)^4 + (1/5)(x-1)^5$
$T_5(3/2) = (1/2) - (1/8) + (1/24) - (1/64) + 1/160$

(b) Estimate the error $|\ln(3/2) - T_5(3/2)|$, using Taylor's Error Bound.

The 6-th derivative is $120/x^6$ and on the interval $[1, 3/2]$ this is bounded by $K = 120$. So the error bound is $(120/6!)(1/2)^6 = (1/6)(1/64) = 1/384$.

(c) The number $\ln(3/2)$ can be written *exactly* as the alternating sum

$$\ln(3/2) = \frac{(1/2)}{1} - \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} - \frac{(1/2)^6}{6} + \cdots .$$

Use this fact to find another estimate for the error $|\ln(3/2) - T_5(3/2)|$.

The next term in the alternating series would be $-(1/6)(1/2)^6 = 1/384$, and its absolute value is the error bound.

(d) Which of these two estimates is the better one?

They are the same. In part (b), the bound $K = 120$ comes from the 6th derivative at 1 (as it happens), so it is not only numerically the same, but computed in the same way.

(Note: The error is actually about .0018.)

(14) Limits of sequences Evaluate these limits:

$(a)\ \lim_{n\to\infty}\left(\frac{\sin(n)}{n}\right)$ $\quad (b)\ \lim_{n\to\infty}(3n)^{1/n}$ $\quad (c)\ \lim_{n\to\infty} n(\sin(1/n))$

$(d)\ \lim_{n\to\infty} n^2(1-\cos(1/n)))$ $\quad (e)\ \lim_{n\to\infty}\left(1-\frac{5}{n}\right)^n$

(a) 0, by the Squeeze Theorem: $\left|\frac{\sin(n)}{n}\right| < 1/n$

(b) 1, since the logarithm $(1/n)\ln(3n)$ goes to 0 with n large (l'Hôpital).

(c) 1, since $\lim_{x\to -+}\lim_{x\to 0^+}\frac{\sin x}{x} = 1$ (from Calc1, or l'Hôpital)

(d) 1, since $\lim_{n\to\infty}\frac{1-\cos(1/n)}{1/n^2} = \lim_{x\to 0^+}\frac{1-\cos x}{x^2} = 1/2$ (Calc1, or l'Hôpital)

(e) e^{-5}, by taking logarithms and applying l'Hôpital.

(15) The sum of an infinite Series

Prove that each of the following series converges, and calculate the sum.

$(a)\ \sum_{n=2}^{\infty}\frac{3^n+(-5)^n}{7^n}$ $\quad (b)\ \sum_{n=4}^{\infty}\frac{1}{n(n+1)}$ $\quad (c)\ \sum_{n=4}^{\infty}\frac{1}{n(n+2)}$

Convergence: The first sum is a sum of two geometric series with ratios less than 1 in absolute value, the 2nd and 3rd can be compared to a p-series with $p = 2$. You could also compute the partial sums explicitly, and check convergence directly.

Values:

$(a)(3/7)^2\frac{1}{1-(3/7)} + (-5/7)^2\frac{1}{1-(-5/7)} = 13/21.$

(b) $1/20+1/30+1/42+\cdots = (1/4-1/5)+(1/5-1/6)+(1/6-1/7)+\cdots = 1/4$

(c) $1/24 + 1/35 + 1/48 + \cdots = (1/2)[(1/4 - 1/6) + (1/5 - 1/7) + (1/6 - 1/8) + (1/7 - 1/9) + \cdots = (1/2)(1/4 + 1/5) = 9/40.$

Convergence via Partial sums

(a) $13/21 - [(3/4)(3/7)^n - (5/12)(-5/7)^n \to 13/21$

(b) $1/4 - 1/(n+1) \to 1/4$

(c) $1/2([1/4 + 1/5] - [(1/(n+1) + 1/(n+2)] \to 1/2[1/4 + 1/5].$

(16) Geometric Series

(a) A student applies the formula for the sum of a geometric series as follows: $1 + 2 + 2^2 + 2^3 + 2^4 + \cdots = \dfrac{1}{1-2} = -1$. What went wrong?

Failure to check the convergence condition for the series: we do not have $|2| < 1$*!*

(b) Applying the formula twice more, the same student calculates:

$$\begin{aligned} 1 - 1 + 1 - 1 + 1 \mp \cdots &= \sum_{n=0}^{\infty} (-1)^n = \frac{1}{1-(-1)} = 1/2 \\ .99999\ldots &= .9 \cdot \sum_{n=0}^{\infty} (.1)^n = .9 \cdot \frac{1}{1-(.1)} = 1 \end{aligned}$$

Are these two conclusions correct?

As $|.9| < 1$*, the second conclusion is correct.*

(c) Write the repeating decimal 0.297297297... as a fraction.

This is $.297 \sum_{n=0}^{\infty} (.001)^n$. As $.001 < 1$, it sums to $.297 \frac{1}{1-.001} = 297/999$ or $11/37$

(17) The Limit Comparison Test

Use the Limit Comparison Test to determine which of the following converges.

(a) $\sum_{n=5}^{\infty} (1 - 5/n)^n$ (b) $\sum_{n=5}^{\infty} \sin(1/n)$ (c) $\sum_{n=5}^{\infty} (1 - \cos(1/n))$

(a) The series diverges by the Divergence Test: Problem 14(e).

(b) The series diverges by comparison with the harmonic series: Problem 14(a).

(c) The series converges by the p-Test with $p = 2$: Problem 14(d).

(The sum is approximately 0.778758796.)

(18) Summation: Error Estimates

Estimate the error involved in taking the sum of the first 10 terms of each of the following series as an approximation to the full sum.

(a) $\sum_{n=1}^{\infty} \dfrac{1}{n^2+n}$ (b) $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^2+n}$

(a) The integral estimate: $\displaystyle\int_{10}^{\infty} \frac{1}{x^2+x} dx = \ln(1.1) < .1$

(b) The error estimate for this type of alternating series is just the absolute value of the next term, $1/(11^2 + 11) = 1/132$ (much closer).

On the other hand: we can sum the series explicitly and get the error exactly.

In (a), the "tail" $\displaystyle\sum_{n=11}^{\infty} \frac{1}{n^2+n}$ is $1/11$, which is easier to calculate and more accurate than the error estimate we found above.

And in (b), the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$ "telescopes" down to $2\displaystyle\sum_{n=1}^{\infty}(-1)^n/n + 1 = -(2\ln 2) + 1$, so we could also calculate the error exactly here. The exact value is uninteresting, but written as a decimal $(.0041\ldots)$ it is informative.

(19) Absolute vs. Conditional Convergence

Define *absolute convergence*, and give an example of a series which converges conditionally, but does not converge absolutely.

A series $\sum_{n=1}^{\infty} a_n$ converges *absolutely* if the corresponding series of non-negative terms $\sum_{n=1}^{\infty} |a_n|$ converges.

The standard example of a convergent series which is not absolutely convergent is the *alternating harmonic series* $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}/n$.

Comments:

The sum of the first n *terms of the harmonic series is very approximately* $\ln(n)$ *(diverging to* ∞*).*

The partial sums of the alternating harmonic series converge slowly to $\ln(2)$*, too slowly to be useful: it takes about 1000 terms to get just 3 digits accuracy.*

If we want to compute $\ln(2)$ *by a power series, we can use the Taylor expansion of* $\ln(x)$ *around* $a = 1$ *taking* $x = 1/2$ *rather than* $x = 2$*, getting a rapidly convergent expression for* $\ln(1/2) = -\ln(2)$.

The point: $1/2$ *is closer to* 1 *than* 2 *is.*