Notes on the Review Problems for Midterm 2

(1)(a) The partial fractions expansion $\dfrac{1}{(x-3)(x-4)} = \dfrac{1}{x-4} - \dfrac{1}{x-3}$ gives

$$\int_5^\infty \frac{dx}{(x-3)(x-4)} = \lim_{b\to\infty} \left(\ln|x-4| - \ln|x-3|\right)\Big|_5^b = \lim_{b\to\infty} \ln\left(\frac{b-4}{b-3}\right) + \ln 2 = \ln 2.$$

(1)(b) L'Hôpital's Rule lets us write

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0.$$

This can be rewritten $\lim\limits_{a\to 0+} a\ln a = 0$. Now we get

$$\int_0^1 \ln x\, dx = \lim_{a\to 0+} \int_a^1 \ln x\, dx = \lim_{a\to 0+} (x\ln x - x)\Big|_a^1 = -1.$$

(1)(c) Integration by parts gives $\displaystyle\int x^n e^{-x}\, dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}\, dx$. Now the fact $\lim\limits_{b\to\infty} b^n e^{-b} = 0$ (which we get from l'Hôpital's Rule) lets us write

$$\int_0^\infty x^n e^{-x}\, dx = n\int_0^\infty x^{n-1}e^{-x}\, dx.$$

Therefore, $\displaystyle\int_0^\infty x^3 e^{-x}\, dx = 3\int_0^\infty x^2 e^{-x}\, dx = 6\int_0^\infty x^1 e^{-x}\, dx = 6\int_0^\infty e^{-x}\, dx = 6.$

(1)(d) The substitution $x = 3\tan\theta$ gives

$$\int \frac{dx}{9+x^2} = \int \frac{3\sec^2\theta\, d\theta}{9(1+\tan^2\theta)} = \int \frac{d\theta}{3} = \frac{\theta}{3} + C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Therefore, $\displaystyle\int_{-\infty}^\infty \frac{dx}{9+x^2} = \lim_{b\to\infty}\left(\frac{1}{3}\tan^{-1}\left(\frac{b}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{-b}{3}\right)\right) = \frac{1}{3}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{3}.$

(2)(a) For $x \ge 7$ we know $0 < x - |\cos x| \le x$, hence $\dfrac{1}{x - |\cos x|} \ge \dfrac{1}{x} > 0$. The divergence of $\displaystyle\int_7^\infty \frac{dx}{x}$ implies the divergence of $\displaystyle\int_7^\infty \frac{dx}{x - |\cos x|}$.

(2)(b) For $x \ge 5$ we know $0 < \dfrac{1}{e^{x^2}} < \dfrac{1}{e^x}$. The convergence of $\displaystyle\int_5^\infty \frac{dx}{e^x}$ (which is just $\displaystyle\int_5^\infty e^{-x}\, dx$) implies the convergence of $\displaystyle\int_5^\infty \frac{dx}{e^{x^2}}$.

(3) The length is

$$\int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1-\cos\theta)^2 + \sin^2\theta\, d\theta} = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$
$$= 2\int_0^{2\pi} \sqrt{\frac{1-\cos\theta}{2}}\, d\theta = 2\int_0^{2\pi} \sqrt{\sin^2(\theta/2)}\, d\theta$$
$$= 2\int_0^{2\pi} |\sin(\theta/2)|\, d\theta = 2\int_0^{2\pi} \sin(\theta/2)\, d\theta = 8.$$

(4) The area is $\frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\, d\theta = \frac{1}{2}(2\pi - 0 + \pi) = \frac{3\pi}{2}$.

(5) The substitution $u = 1 + \frac{9}{4}(x+2)$ leads to

$$\text{length } = \int_0^1 \sqrt{1 + \frac{9}{4}(x+2)}\, dx = \frac{8}{27}\left(\frac{11}{2} + \frac{9x}{4}\right)^{3/2}\Bigg|_0^1 .$$

(6) A sphere with radius R is obtained by rotating the semicircle $y = \sqrt{R^2 - x^2}$, $-R \leq x \leq R$ about the x-axis. In this case,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2} = \sqrt{\frac{R^2}{R^2 - x^2}} = \frac{R}{\sqrt{R^2 - x^2}}.$$

The surface area is $2\pi \int_{-R}^{R} \sqrt{R^2 - x^2} \cdot \frac{R}{\sqrt{R^2 - x^2}}\, dx = 4\pi R^2$.

(7) Multiplying $r = \sin\theta$ by r, we get $r^2 = r\sin\theta$, which is $x^2 + y^2 = y$. This is $(x-0)^2 + (y - 1/2)^2 = (1/2)^2$. The center of the circle is $(0, 1/2)$. The radius of the circle is $1/2$.

(8) We can use $x = \frac{6\cos t}{3}$, $y = \frac{6\sin t}{4}$, $0 \leq t \leq 2\pi$.

(9) The length is $\int_1^2 \sqrt{4t^2 + 9t^4}\, dt = \int_1^2 t\sqrt{4 + 9t^2}\, dt$. We can compute this integral using the substitution $u = 4 + 9t^2$.

(10)(a) $\frac{dx}{dt} = \tan x$ leads to $\int \frac{\cos x\, dx}{\sin x} = \int 1\, dt$, $\ln|\sin x| = t + C$, $|\sin x| = e^t e^C = Be^t$, where $B = e^C$ is a constant. Now we get $\sin x = \pm Be^t = Ae^t$, where A is a constant. Substituting $t = 0$ and $x = \pi/6$, we get $1/2 = A$. The initial value problem is solved by $x = \sin^{-1}((1/2)e^t)$.

(10)(b) $(4+x^3)^{1/2}\frac{dy}{dx} = (xy)^2$ leads to $\int \frac{dy}{y^2} = \int \frac{x^2}{(4+x^3)^{1/2}}\, dx$, $-\frac{1}{y} = \frac{2}{3}(4+x^3)^{1/2} + C$. Substituting $x = 0$ and $y = -1$, we get $C = -\frac{1}{3}$. The initial value problem is solved by $y = \frac{3}{1 - 2\sqrt{4 + x^3}}$.

(11) The temperature T of the coffee is given by $T = 25 + Ce^{-kt}$. The initial condition $T(0) = 50$ implies $C = 25$. Now we know that the temperature T of the coffee is given by $T = 25 + 25e^{-kt}$. Substituting $t = 1$, we get $30 = 25 + 25e^{-k}$, which implies $k = \ln 5$. The problem asks us to find t such that $40 = 25 + 25e^{-kt}$. This last equation gives $e^{-kt} = 3/5$, hence $-kt = \ln(3/5)$. We solve for t using $k = \ln 5$.

(12) We know $P(t) = \frac{20,000}{.05} + Ce^{(.05)t}$ and $P(30) = 0$. This gives us $C = -(400,000)e^{-3/2}$. The initial balance is $P(0) = 400,000 - (400,000)e^{-3/2}$.

(13) Since $f(x) = x^{-1}$, we get $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$, $f'''(x) = -6x^{-4}$. This implies $f(1) = 1$, $f'(1) = -1$, $f''(1) = 2$, $f'''(1) = -6$. Now we know

$$\begin{aligned} T_3(x) &= 1 + (-1)(x-1) + \frac{2(x-1)^2}{2} + \frac{-6(x-1)^3}{6} \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3. \end{aligned}$$

We know $|f^{(4)}(u)| = 24|u^{-5}| \leq 24$ when $1 \leq u \leq 3/2$. This says that we can use $K = 24$. This implies

$$|f(3/2) - T_3(3/2)| \leq \frac{24|3/2 - 1|^4}{4!} = \frac{1}{16}.$$

(14)(a) The inequalities $-1 \leq \sin n \leq 1$ lead to $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. Since $\lim_{n\to\infty} -\frac{1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$, the Squeeze Theorem implies $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

(14)(b) L'Hôpital's Rule gives $\lim_{x\to\infty} \ln\left((3x)^{1/x}\right) = \lim_{x\to\infty} \frac{\ln(3x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$. Exponentiating this, we get $\lim_{x\to\infty} (3x)^{1/x} = e^0 = 1$. This implies $\lim_{n\to\infty} (3n)^{1/n} = 1$.

(14)(c) L'Hôpital's Rule gives

$$\lim_{x\to\infty} \ln\left(\left(1 - \frac{5}{x}\right)^x\right) = \lim_{x\to\infty} x \ln\left(1 - \frac{5}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{5}{x}\right)}{1/x} = \lim_{x\to\infty} \frac{\left(\frac{5/x^2}{1-5/x}\right)}{-1/x^2} = -5.$$

Exponentiating this, we get $\lim_{x\to\infty} \left(1 - \frac{5}{x}\right)^x = e^{-5}$. This implies $\lim_{n\to\infty} \left(1 - \frac{5}{n}\right)^n = e^{-5}$.

(14)(d) Since $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\lim_{n\to\infty} 1/n = 0$, we conclude $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$. This is equivalent to $\lim_{n\to\infty} n\sin(1/n) = 1$.

(14)(e) L'Hôpital's Rule gives $\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}$. Since $\lim_{n\to\infty} 1/n = 0$, we conclude $\lim_{n\to\infty} \frac{1 - \cos(1/n)}{(1/n)^2} = \frac{1}{2}$. This is equivalent to $\lim_{n\to\infty} n^2(1 - \cos(1/n)) = \frac{1}{2}$.

(15) Since $|\frac{1}{1000}| < 1$, the formula for the sum of a geometric series gives

$$\begin{aligned} 5.273273273\ldots &= 5 + \frac{273}{1000} + \frac{273}{(1000)^2} + \frac{273}{(1000)^3} + \cdots \\ &= 5 + \frac{273}{1000}\left(1 + \frac{1}{1000} + \left(\frac{1}{1000}\right)^2 + \left(\frac{1}{1000}\right)^3 + \cdots\right) \\ &= 5 + \frac{273}{1000}\left(\frac{1}{1-\frac{1}{1000}}\right) = 5 + \frac{273}{999} = \frac{5268}{999}\ . \end{aligned}$$

(16)(a) $\displaystyle\sum_{n=3}^{\infty} \frac{2^n}{3^{n+1}} = \frac{2^3}{3^4}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right) = \frac{2^3}{3^4} \cdot \frac{1}{1-2/3} = \frac{8}{27}.$

(16)(b) $\displaystyle\sum_{n=4}^{N} \frac{1}{n(n-1)} = \sum_{n=4}^{N}\left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{3} - \frac{1}{N}$ because the other terms cancel out in pairs. Now

$$\sum_{n=4}^{\infty} \frac{1}{n(n-1)} = \lim_{N\to\infty} \sum_{n=4}^{N} \frac{1}{n(n-1)} = \lim_{N\to\infty}\left(\frac{1}{3} - \frac{1}{N}\right) = \frac{1}{3}.$$

(17)(a) Since $\dfrac{1}{\sqrt{n}}$ is decreasing and approaches 0, the Leibniz Test tells us that $\displaystyle\sum_{n=5}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges.

(17)(b) $\displaystyle\int_5^{\infty} \frac{dx}{x(\ln x)} = \lim_{b\to\infty} \ln(\ln x)\Big|_5^b = \infty$, hence $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n(\ln n)}$ diverges by the Integral Test.

(17)(c) $\displaystyle\int_5^{\infty} \frac{dx}{x(\ln x)^{3/2}} = \lim_{b\to\infty} -2(\ln x)^{-1/2}\Big|_5^b = 2(\ln 5)^{-1/2} < \infty$, hence $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n(\ln n)^{3/2}}$ converges by the Integral Test.

(17)(d) Since the answer to 14(c) is $\displaystyle\lim_{n\to\infty}\left(1 - \frac{5}{n}\right)^n = e^{-5} \neq 0$, the Test For Divergence says that $\displaystyle\sum_{n=4}^{\infty}\left(1 - \frac{5}{n}\right)^n$ diverges.

(17)(e) We do a limit comparison with the series $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n}$. The answer to 14(d) says $\displaystyle\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1$. Since this limit is positive and finite, we conclude divergence of $\displaystyle\sum_{n=4}^{\infty} \sin(1/n)$ from the divergence of $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n}$.

(17)(f) We do a limit comparison with the series $\sum_{n=4}^{\infty} \frac{1}{n^2}$. The answer to 14(e) says $\lim_{n\to\infty} \frac{1-\cos(1/n)}{1/n^2} = \frac{1}{2}$. Since this limit is positive and finite, we conclude convergence of $\sum_{n=4}^{\infty} \big(1-\cos(1/n)\big)$ from the convergence of $\sum_{n=4}^{\infty} \frac{1}{n^2}$.

(17)(g) Since $|2/3| < 1$, we know that $\sum_{n=2}^{\infty} \frac{2^n}{3^n} = \sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n$ converges. The Comparison Test and $0 < \frac{2^n}{3^n+1} < \frac{2^n}{3^n}$ allow us to conclude that $\sum_{n=2}^{\infty} \frac{2^n}{3^n+1}$ converges.

(17)(h) The Limit Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is successful because

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^4-n^3-4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{n^4-n^3-4} = 1,$$

which is a positive and finite limit. Since the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, we conclude that the series $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n^3-4}$ converges.

(18)(a) If we take the sum of only the first 10 terms of $\sum_{n=1}^{\infty} \frac{1}{n^2}$, then the absolute value of the error is at most $\int_{10}^{\infty} \frac{dx}{x^2} = \frac{1}{10}$.

(18)(b) If we take the sum of only the first 10 terms of $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, then the absolute value of the error is at most the absolute value of the first omitted term, which is $\frac{1}{11^2}$.

(19) An infinite series $\sum a_n$ converges absolutely when $\sum |a_n|$ converges. An infinite series $\sum a_n$ converges conditionally when it converges, but does not converge absolutely. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.

(20) Let L denote $\lim_{n\to\infty} a_n$. Taking the limit in the equation $a_{n+1} = \sqrt{12+a_n}$, we get $L = \sqrt{12+L}$. The number L must be a solution of the equation $L^2 = 12 + L$. This means that L must be either 4 or -3. The equation $L = \sqrt{12+L}$ excludes the possibility $L = -3$. We must have $L = 4$.