

## Notes on the Review Problems for Midterm 2

(1)(a) The partial fractions expansion  $\frac{1}{(x-3)(x-4)} = \frac{1}{x-4} - \frac{1}{x-3}$  gives

$$\int_5^\infty \frac{dx}{(x-3)(x-4)} = \lim_{b \rightarrow \infty} (\ln|x-4| - \ln|x-3|) \Big|_5^b = \lim_{b \rightarrow \infty} \ln \left( \frac{b-4}{b-3} \right) + \ln 2 = \ln 2.$$

(1)(b) L'Hôpital's Rule lets us write

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0.$$

This can be rewritten  $\lim_{a \rightarrow 0^+} a \ln a = 0$ . Now we get

$$\int_0^1 \ln x \, dx = \lim_{a \rightarrow 0^+} \int_a^1 \ln x \, dx = \lim_{a \rightarrow 0^+} (x \ln x - x) \Big|_a^1 = -1.$$

(1)(c) Integration by parts gives  $\int x^n e^{-x} \, dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} \, dx$ . Now the fact  $\lim_{b \rightarrow \infty} b^n e^{-b} = 0$  (which we get from l'Hôpital's Rule) lets us write

$$\int_0^\infty x^n e^{-x} \, dx = n \int_0^\infty x^{n-1} e^{-x} \, dx.$$

Therefore,  $\int_0^\infty x^3 e^{-x} \, dx = 3 \int_0^\infty x^2 e^{-x} \, dx = 6 \int_0^\infty x e^{-x} \, dx = 6 \int_0^\infty e^{-x} \, dx = 6$ .

(1)(d) The substitution  $x = 3 \tan \theta$  gives

$$\int \frac{dx}{9+x^2} = \int \frac{3 \sec^2 \theta \, d\theta}{9(1+\tan^2 \theta)} = \int \frac{d\theta}{3} = \frac{\theta}{3} + C = \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) + C.$$

Therefore,  $\int_{-\infty}^\infty \frac{dx}{9+x^2} = \lim_{b \rightarrow \infty} \left( \frac{1}{3} \tan^{-1} \left( \frac{b}{3} \right) - \frac{1}{3} \tan^{-1} \left( \frac{-b}{3} \right) \right) = \frac{1}{3} \left( \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) \right) = \frac{\pi}{3}$ .

(2)(a) For  $x \geq 7$  we know  $0 < x - |\cos x| \leq x$ , hence  $\frac{1}{x - |\cos x|} \geq \frac{1}{x} > 0$ . The divergence of  $\int_7^\infty \frac{dx}{x}$  implies the divergence of  $\int_7^\infty \frac{dx}{x - |\cos x|}$ .

(2)(b) For  $x \geq 5$  we know  $0 < \frac{1}{e^{x^2}} < \frac{1}{e^x}$ . The convergence of  $\int_5^\infty \frac{dx}{e^x}$  (which is just  $\int_5^\infty e^{-x} \, dx$ ) implies the convergence of  $\int_5^\infty \frac{dx}{e^{x^2}}$ .

(3) The length is

$$\begin{aligned}\int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta &= \int_0^{2\pi} \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} d\theta = \int_0^{2\pi} \sqrt{2 - 2 \cos \theta} d\theta \\ &= 2 \int_0^{2\pi} \sqrt{\frac{1 - \cos \theta}{2}} d\theta = 2 \int_0^{2\pi} \sqrt{\sin^2(\theta/2)} d\theta \\ &= 2 \int_0^{2\pi} |\sin(\theta/2)| d\theta = 2 \int_0^{2\pi} \sin(\theta/2) d\theta = 8.\end{aligned}$$

(4) The area is  $\frac{1}{2} \int_0^{2\pi} (1 - \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} 1 - 2 \cos \theta + \cos^2 \theta d\theta = \frac{1}{2}(2\pi - 0 + \pi) = \frac{3\pi}{2}$ .

(5) The substitution  $u = 1 + \frac{9}{4}(x + 2)$  leads to

$$\text{length} = \int_0^1 \sqrt{1 + \frac{9}{4}(x + 2)} dx = \frac{8}{27} \left( \frac{11}{2} + \frac{9x}{4} \right)^{3/2} \Big|_0^1.$$

(6) A sphere with radius  $R$  is obtained by rotating the semicircle  $y = \sqrt{R^2 - x^2}$ ,  $-R \leq x \leq R$  about the  $x$ -axis. In this case,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{R^2 - x^2}}\right)^2} = \sqrt{\frac{R^2}{R^2 - x^2}} = \frac{R}{\sqrt{R^2 - x^2}}.$$

The surface area is  $2\pi \int_{-R}^R \sqrt{R^2 - x^2} \cdot \frac{R}{\sqrt{R^2 - x^2}} dx = 4\pi R^2$ .

(7) Multiplying  $r = \sin \theta$  by  $r$ , we get  $r^2 = r \sin \theta$ , which is  $x^2 + y^2 = y$ . This is  $(x - 0)^2 + (y - 1/2)^2 = (1/2)^2$ . The center of the circle is  $(0, 1/2)$ . The radius of the circle is  $1/2$ .

(8) We can use  $x = \frac{6 \cos t}{3}$ ,  $y = \frac{6 \sin t}{4}$ ,  $0 \leq t \leq 2\pi$ .

(9) The length is  $\int_1^2 \sqrt{4t^2 + 9t^4} dt = \int_1^2 t \sqrt{4 + 9t^2} dt$ . We can compute this integral using the substitution  $u = 4 + 9t^2$ .

(10)(a)  $\frac{dx}{dt} = \tan x$  leads to  $\int \frac{\cos x dx}{\sin x} = \int 1 dt$ ,  $\ln |\sin x| = t + C$ ,  $|\sin x| = e^t e^C = B e^t$ , where  $B = e^C$  is a constant. Now we get  $\sin x = \pm B e^t = A e^t$ , where  $A$  is a constant. Substituting  $t = 0$  and  $x = \pi/6$ , we get  $1/2 = A$ . The initial value problem is solved by  $x = \sin^{-1}((1/2)e^t)$ .

(10)(b)  $(4 + x^3)^{1/2} \frac{dy}{dx} = (xy)^2$  leads to  $\int \frac{dy}{y^2} = \int \frac{x^2}{(4 + x^3)^{1/2}} dx$ ,  $-\frac{1}{y} = \frac{2}{3}(4 + x^3)^{1/2} + C$ .

Substituting  $x = 0$  and  $y = -1$ , we get  $C = -\frac{1}{3}$ . The initial value problem is solved by

$$y = \frac{3}{1 - 2\sqrt{4 + x^3}}.$$

(11) The temperature  $T$  of the coffee is given by  $T = 25 + Ce^{-kt}$ . The initial condition  $T(0) = 50$  implies  $C = 25$ . Now we know that the temperature  $T$  of the coffee is given by  $T = 25 + 25e^{-kt}$ . Substituting  $t = 1$ , we get  $30 = 25 + 25e^{-k}$ , which implies  $k = \ln 5$ . The problem asks us to find  $t$  such that  $40 = 25 + 25e^{-kt}$ . This last equation gives  $e^{-kt} = 3/5$ , hence  $-kt = \ln(3/5)$ . We solve for  $t$  using  $k = \ln 5$ .

(12) We know  $P(t) = \frac{20,000}{.05} + Ce^{(.05)t}$  and  $P(30) = 0$ . This gives us  $C = -(400,000)e^{-3/2}$ . The initial balance is  $P(0) = 400,000 - (400,000)e^{-3/2}$ .

(13) Since  $f(x) = x^{-1}$ , we get  $f'(x) = -x^{-2}$ ,  $f''(x) = 2x^{-3}$ ,  $f'''(x) = -6x^{-4}$ . This implies  $f(1) = 1$ ,  $f'(1) = -1$ ,  $f''(1) = 2$ ,  $f'''(1) = -6$ . Now we know

$$\begin{aligned} T_3(x) &= 1 + (-1)(x-1) + \frac{2(x-1)^2}{2} + \frac{-6(x-1)^3}{6} \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3. \end{aligned}$$

We know  $|f^{(4)}(u)| = 24|u^{-5}| \leq 24$  when  $1 \leq u \leq 3/2$ . This says that we can use  $K = 24$ . This implies

$$|f(3/2) - T_3(3/2)| \leq \frac{24|3/2 - 1|^4}{4!} = \frac{1}{16}.$$

(14)(a) The inequalities  $-1 \leq \sin n \leq 1$  lead to  $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ . Since  $\lim_{n \rightarrow \infty} -\frac{1}{n} = 0 = \lim_{n \rightarrow \infty} \frac{1}{n}$ , the Squeeze Theorem implies  $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$ .

(14)(b) L'Hôpital's Rule gives  $\lim_{x \rightarrow \infty} \ln((3x)^{1/x}) = \lim_{x \rightarrow \infty} \frac{\ln(3x)}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0$ . Exponentiating this, we get  $\lim_{x \rightarrow \infty} (3x)^{1/x} = e^0 = 1$ . This implies  $\lim_{n \rightarrow \infty} (3n)^{1/n} = 1$ .

(14)(c) L'Hôpital's Rule gives

$$\lim_{x \rightarrow \infty} \ln \left( \left( 1 - \frac{5}{x} \right)^x \right) = \lim_{x \rightarrow \infty} x \ln \left( 1 - \frac{5}{x} \right) = \lim_{x \rightarrow \infty} \frac{\ln \left( 1 - \frac{5}{x} \right)}{1/x} = \lim_{x \rightarrow \infty} \frac{\left( \frac{5/x^2}{1-5/x} \right)}{-1/x^2} = -5.$$

Exponentiating this, we get  $\lim_{x \rightarrow \infty} \left( 1 - \frac{5}{x} \right)^x = e^{-5}$ . This implies  $\lim_{n \rightarrow \infty} \left( 1 - \frac{5}{n} \right)^n = e^{-5}$ .

(14)(d) Since  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$  and  $\lim_{n \rightarrow \infty} 1/n = 0$ , we conclude  $\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = 1$ . This is equivalent to  $\lim_{n \rightarrow \infty} n \sin(1/n) = 1$ .

(14)(e) L'Hôpital's Rule gives  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{2x} = \frac{1}{2}$ . Since  $\lim_{n \rightarrow \infty} 1/n = 0$ , we conclude  $\lim_{n \rightarrow \infty} \frac{1 - \cos(1/n)}{(1/n)^2} = \frac{1}{2}$ . This is equivalent to  $\lim_{n \rightarrow \infty} n^2(1 - \cos(1/n)) = \frac{1}{2}$ .

(15) Since  $|\frac{1}{1000}| < 1$ , the formula for the sum of a geometric series gives

$$\begin{aligned} 5.273273273\dots &= 5 + \frac{273}{1000} + \frac{273}{(1000)^2} + \frac{273}{(1000)^3} + \dots \\ &= 5 + \frac{273}{1000} \left( 1 + \frac{1}{1000} + \left(\frac{1}{1000}\right)^2 + \left(\frac{1}{1000}\right)^3 + \dots \right) \\ &= 5 + \frac{273}{1000} \left( \frac{1}{1 - \frac{1}{1000}} \right) = 5 + \frac{273}{999} = \frac{5268}{999}. \end{aligned}$$

(16)(a)  $\sum_{n=3}^{\infty} \frac{2^n}{3^{n+1}} = \frac{2^3}{3^4} \left( 1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \dots \right) = \frac{2^3}{3^4} \cdot \frac{1}{1 - 2/3} = \frac{8}{27}.$

(16)(b)  $\sum_{n=4}^N \frac{1}{n(n-1)} = \sum_{n=4}^N \left( \frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{3} - \frac{1}{N}$  because the other terms cancel out in pairs. Now

$$\sum_{n=4}^{\infty} \frac{1}{n(n-1)} = \lim_{N \rightarrow \infty} \sum_{n=4}^N \frac{1}{n(n-1)} = \lim_{N \rightarrow \infty} \left( \frac{1}{3} - \frac{1}{N} \right) = \frac{1}{3}.$$

(17)(a) Since  $\frac{1}{\sqrt{n}}$  is decreasing and approaches 0, the Leibniz Test tells us that  $\sum_{n=5}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$  converges.

(17)(b)  $\int_5^{\infty} \frac{dx}{x(\ln x)} = \lim_{b \rightarrow \infty} \ln(\ln x) \Big|_5^b = \infty$ , hence  $\sum_{n=5}^{\infty} \frac{1}{n(\ln n)}$  diverges by the Integral Test.

(17)(c)  $\int_5^{\infty} \frac{dx}{x(\ln x)^{3/2}} = \lim_{b \rightarrow \infty} -2(\ln x)^{-1/2} \Big|_5^b = 2(\ln 5)^{-1/2} < \infty$ , hence  $\sum_{n=5}^{\infty} \frac{1}{n(\ln n)^{3/2}}$  converges by the Integral Test.

(17)(d) Since the answer to 14(c) is  $\lim_{n \rightarrow \infty} \left( 1 - \frac{5}{n} \right)^n = e^{-5} \neq 0$ , the Test For Divergence says that  $\sum_{n=4}^{\infty} \left( 1 - \frac{5}{n} \right)^n$  diverges.

(17)(e) We do a limit comparison with the series  $\sum_{n=4}^{\infty} \frac{1}{n}$ . The answer to 14(d) says

$\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = 1$ . Since this limit is positive and finite, we conclude divergence of  $\sum_{n=4}^{\infty} \sin(1/n)$  from the divergence of  $\sum_{n=4}^{\infty} \frac{1}{n}$ .

(17)(f) We do a limit comparison with the series  $\sum_{n=4}^{\infty} \frac{1}{n^2}$ . The answer to 14(e) says

$\lim_{n \rightarrow \infty} \frac{1 - \cos(1/n)}{1/n^2} = \frac{1}{2}$ . Since this limit is positive and finite, we conclude convergence of  $\sum_{n=4}^{\infty} (1 - \cos(1/n))$  from the convergence of  $\sum_{n=4}^{\infty} \frac{1}{n^2}$ .

(17)(g) Since  $|2/3| < 1$ , we know that  $\sum_{n=2}^{\infty} \frac{2^n}{3^n} = \sum_{n=2}^{\infty} \left(\frac{2}{3}\right)^n$  converges. The Comparison

Test and  $0 < \frac{2^n}{3^n + 1} < \frac{2^n}{3^n}$  allow us to conclude that  $\sum_{n=2}^{\infty} \frac{2^n}{3^n + 1}$  converges.

(17)(h) The Limit Comparison Test with  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  is successful because

$$\lim_{n \rightarrow \infty} \frac{\frac{n^2}{n^4 - n^3 - 4}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^4}{n^4 - n^3 - 4} = 1,$$

which is a positive and finite limit. Since the series  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  converges, we conclude that

the series  $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - n^3 - 4}$  converges.

(18)(a) If we take the sum of only the first 10 terms of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ , then the absolute value of

the error is at most  $\int_{10}^{\infty} \frac{dx}{x^2} = \frac{1}{10}$ .

(18)(b) If we take the sum of only the first 10 terms of  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ , then the absolute value

of the error is at most the absolute value of the first omitted term, which is  $\frac{1}{11^2}$ .

(19) An infinite series  $\sum a_n$  converges absolutely when  $\sum |a_n|$  converges. An infinite series  $\sum a_n$  converges conditionally when it converges, but does not converge absolutely. The

series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  converges conditionally.

(20) Let  $L$  denote  $\lim_{n \rightarrow \infty} a_n$ . Taking the limit in the equation  $a_{n+1} = \sqrt{12 + a_n}$ , we get  $L = \sqrt{12 + L}$ . The number  $L$  must be a solution of the equation  $L^2 = 12 + L$ . This means that  $L$  must be either 4 or  $-3$ . The equation  $L = \sqrt{12 + L}$  excludes the possibility  $L = -3$ . We must have  $L = 4$ .