Remember that most of the material on the final exam is reviewed in the first two review sheets. This sheet only deals with the most recent material.

Additional Review Problems on Sequences and Series

1. Interval of Convergence
Find the precise interval of convergence for each of the series
$(a)\ \sum_{n=1}^{\infty} \frac{(x-3)^n}{(n+1)2^n} \qquad (b)\ \sum_{n=1}^{\infty} \frac{(-1)^n(x+3)^n}{\sqrt{n+1}}$

(a) By the ratio or root test, we require $\left|\frac{x-3}{2}\right| \leq 1$. So the series converges absolutely for $|x-3| < 2$, $x \in (1,5)$. At the endpoints $x = 1$, $x = 5$ the series becomes an alternating harmonic series or an ordinary harmonic series so it converges on $[1,5)$ (*conditionally* at $x = 1$).

(b) By the ratio or root test, we require $|x+3| \leq 1$. So the series converges absolutely for $|x+3| < 1$, $x \in (-4,-2)$. At $x = -2$ the series converges by the Leibniz test (conditionally), while at $x = -4$ it is a p-series with $p = 1/2$, so divergent. The interval of convergence is $(-4,-2]$.

2. Interval of Convergence
If there is a power series with the specified interval of convergence, give an example. If not, explain why not.
$(a)\ (-2,1) \quad (b)\ [0,\infty) \quad (c)\ (-1,1]$

We need a symmetric interval, which excludes (b). For (a) and (c) we have examples, based on geometric series (divergent at the endpoints) and p-series (conditionally convergent at one endpoint).

(a) Midpoint $-1/2$, radius $3/2$: $\sum_{n=0}^{\infty} \frac{(x+1/2)^n}{(3/2)^n}$.

(c) Midpoint 0, radius 1: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$.

Problem **3** is on the next page.

3. Tests for Convergence and Divergence

State which tests are used in each case to determine whether the series converges, and what the conclusion is.

$(a)\ \sum_{n=7}^{\infty} \frac{(-1)^n(n+3)}{n+6}$	$(b)\ \sum_{n=2}^{\infty} \sqrt{\frac{n^2+1}{n^5-n^4-2}}$	$(c)\ \sum_{n=1}^{\infty} \sin(1/n)$
$(d)\ \sum_{n=3}^{\infty} \frac{\sin^2(e^n)}{n^{4/3}}$	$(e)\ \sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$	$(f)\ \sum_{n=3}^{\infty} \frac{3n^2-n-1}{5n^3+n^2+5}$
$(g)\ \sum_{n=1}^{\infty} \sin(1/n^2)$	$(h)\ \sum_{n=1}^{\infty} \frac{3^n+n}{4^n-3^n}$	$(i)\ \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4}$
$(j)\ \sum_{n=1}^{\infty} \frac{(2n)!n!}{(3n)!}$	$(k)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$	$(l)\ \sum_{n=3}^{\infty} \frac{\cos(n^3)}{n(\ln n)^2}$
$(m)\ \sum_{n=2}^{\infty} \frac{7^n n^7}{2^{2n-3}}$	$(n)\ \sum_{n=1}^{\infty} \frac{(n!)^2(\pi)^n}{(2n)!}$	$(o)\ \sum_{n=1}^{\infty} (-1)^{n+3}\sqrt{n+3}$

(a, o) Divergent by the *Divergence Test.*

(b, g) Absolutely convergent by *Limit Comparison* with *p-Series* ($p = 3/2$, $p = 2$). The computation $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ is an important part of the solutions of problems (b, g).

(c, f) Divergent by *Limit Comparison* with the *Harmonic Series* ($p = 1$). The computation of $\lim_{n\to\infty} \frac{a_n}{b_n}$ is an important part of the solutions of problems (c, f).

(d) Absolutely convergent by *Comparison* with *p-Series* ($p = 4/3$). You have to write down the inequality that leads to the use of the Comparison Test.

(e) Absolutely convergent by the *Ratio Test:* $\frac{\frac{(n+1)^{n+1}}{n^n}}{3(n+1)} = \left(1+\frac{1}{n}\right)^n/3 \to e/3 < 1$.

(h) Absolutely convergent by *Limit Comparison* with the *Geometric Series* $\sum (3/4)^n$, or by the *Ratio* or *Root Tests* similarly (but less easily). The computation of $\lim_{n\to\infty} \frac{a_n}{b_n}$ is an important part of the use of the Limit Comparison Test.

(i) Absolutely convergent by the *Integral Test.* You have to compute the appropriate improper integral.

(j) Absolutely convergent by the *Ratio Test:* $\frac{(2n+2)(2n+1)(n+1)}{(3n+3)(3n+2)(3n+1)}$ approaches $4/27 < 1$

(k) Convergent by the *Leibniz Test* for alternating series, but only conditionally. The corresponding positive series is a p-*Series* with $p < 1$.

(l) Absolutely convergent by the *Integral Test*, with assistance from the *Comparison Test*. A cousin to (i). You have to write down the details of the uses of these tests.

(m) Divergent by the *Root Test* as $\sqrt[n]{a_n}$ approaches $7/2^2 > 1$. Or by the *Ratio Test*, similarly.

(n) Absolutely convergent by the *Ratio Test:* $\dfrac{(n+1)^2\pi}{(2n+2)(2n+1)}$ aproaches $\pi/4 < 1$.

Problem **4** is on the next page.

4. Algebraic manipulations with Power Series.

(a) The "hyperbolic cosine" ($\cosh(x)$) is defined by $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. Find its Maclaurin expansion. How is this series related to the Maclaurin expansion of $\cos(x)$?

Maclaurin expansion of $\cosh(x)$: $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ (odd degree terms cancel out).

Apart from signs, this is the same as the Maclaurin series for $\cos(x)$. (In terms of the complex number $i = \sqrt{-1}$, we have $\cos(x) = \cosh(ix)$, by substitution.)

(b) Calculate the first 3 nonzero terms of the Maclaurin series for $\tan x$.

Here are the first five:

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \cdots$$

(As $\tan(x)$ is an odd function, only odd powers of x occur here.)

(c) Calculate the first 5 nonzero terms of the Maclaurin series for $x^3 e^{-x^2}$.

Here are the first five:

$$x^3 - x^5 + \frac{x^7}{2} - \frac{x^9}{6} + \frac{x^{11}}{24}$$

5. Evaluation of power series

Let $f(x)$ be defined by $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

(a) Show that $f'(x) = 2f(x)$:

Since $\dfrac{2^n \cdot (nx^{n-1})}{n!} = 2 \cdot \dfrac{2^{n-1}x^{n-1}}{(n-1)!}$, we have

$$f'(x) = 2\sum_{n=1}^{\infty} \frac{2^{n-1}x^{n-1}}{(n-1)!} = 2f(x)$$

(As n goes from 1 to ∞, $n-1$ goes from 0 to ∞.)

(b) Find an explicit formula for $f(x)$:

By (a), f is a solution of $y' = 2y$, so $f(x) = Ce^{2x}$ with $C = f(0) = 1$. So $f(x) = e^{2x}$.

—Or by inspection, $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = e^{2x}$. We don't really need the differential equation here.

6. The Binomial Series Find the first four nonzero terms of the Maclaurin expansion of $f(x) = \sqrt{4 + x^2}$.

$f(x) = 2\left(1 + \dfrac{x^2}{4}\right)^{1/2}$ The first four nonzero terms:

$$2\left[1 + (1/2)\frac{x^2}{4} + \frac{(1/2)(-1/2)}{2}\left(\frac{x^2}{4}\right)^2 + \frac{(1/2)(-1/2)(-3/2)}{6}\left(\frac{x^2}{4}\right)^3\right]$$

Or, $2 + \dfrac{x^2}{4} - \dfrac{x^4}{16} + \dfrac{x^6}{512}$

7. Convergence of Power Series

Use the Maclaurin expansion for e^x to show that $e = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

Take $x = 1$. This is justified by the theorem on functions with $f^{(n)}(x)$ bounded on an interval (§10.7); which follows from Taylor's Error Bound.

8. Evaluating Power Series

Find the Taylor expansion of $\dfrac{1}{(1-x)^2}$ by the following three methods.

(a) Apply the Binomial Theorem; (b) Square the series for $\dfrac{1}{(1-x)}$; and (c) Differentiate the series for $\dfrac{1}{(1-x)}$.

—And rank the methods according to your preference in this case.

You get $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$ by any one of these methods.

(a) Binomial: $\binom{-2}{n} = \frac{(-2)(-3)\cdot\cdots\cdot(-[n+1])}{n!} = (-1)^n(n+1)$ so by the Binomial series with $a = -2$, and with $-x$ in place of x, we get $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$.

(b) Squaring: $(1 + x + x^2 + \cdots)^2 = (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)$. The terms of degree n are $1\cdot x^n + x\cdot x^{n-1} + \cdots + x^{n-1}\cdot x + x^n\cdot 1 = (n+1)x^n$ since there are $n+1$ such terms.

(c) Differentiation: $\displaystyle\left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$

It may be useful to write down the first three terms of each sum to see what these equations are saying.

I'll take option (c). If forced to choose one of the others methods, I'd take (a) as more systematic.

Problem **9** is on the next page.

9. Summation Formulas Calculate $\sum_{n=1}^{\infty} \frac{n}{2^n}$ exactly:

(a) Obtain a simple formula for the sum $f(x) = \sum_{n=1}^{\infty} nx^n$.

Best to do Problem 8 first: $f(x) = x \cdot \sum_{n=1}^{\infty} nx^{n-1} = x \cdot \sum_{n=0}^{\infty}(n+1)x^n = \frac{x}{(1-x)^2}$ by Problem 8.

(b) Check that $x = 1/2$ is in the interval of convergence, and then take $x = 1/2$.

Use the Root test or Ratio test. And then, $f(1/2) = \sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = 2$.

10. Numerical Methods for Integration: Power Series

(a) Use the Maclaurin expansion of degree 3 for $\sin x$ to estimate $\int_0^{1/2} \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$.

$T_3(x) = x - x^3/6$ so we approximate $\frac{\sin(\sqrt{x})}{\sqrt{x}}$ by $1 - x/6$, and the integral by $\left(x - \frac{x^2}{12}\right)\Big|_0^{1/2} = (1/2) - (1/48)$.

(b) Give an estimate for the resulting error.

Our expansion is an alternating series with decreasing terms so we can use the next term as an estimate: the next term is $\int_0^{1/2} \frac{x^2}{5!}\, dx$ or $\frac{(1/2)^3}{3 \cdot 5!} = \frac{1}{24 \cdot 120}$, which is small.

(c) How many (non-zero) terms of the expansion would be needed to meet an error tolerance of 10^{-6}?

With n terms, the error bound is $\int_0^{1/2} \frac{x^n}{(2n+1)!}\, dx = \frac{(1/2)^{n+1}}{(n+1)(2n+1)!}$. Use a calculator to see that 4 terms will do the trick.

Problem **11** is on the next page.

11. Limits

Find $\lim_{n\to\infty}\left[\dfrac{\sin\left(\frac{1}{n}\right)-\left(\frac{1}{n}\right)}{\left(e^{\frac{1}{n}}-1\right)\left(1-\cos\left(\frac{1}{n}\right)\right)}\right]$ using the Maclaurin expansions for sin, cos, and e^x.

We write this as

$$\lim_{x\to 0}\frac{\sin(x)-x}{(e^x-1)(1-\cos(x))}$$

and use the Maclaurin expansions to write $\sin(x)-x\approx -x^3/6$, $e^x-1\approx x$, $1-\cos(x)\approx x^2/2$, so:

$$\frac{-x^3/6}{(x)\cdot\left(\frac{x^2}{2}\right)}=-1/3$$

is the limit.

12. Computation of $\sqrt{5}$ by Newton's Method: Convergence.

In the computation of $\sqrt{5}$ by Newton's Method in your text (§4.8, p. 280), the following sequence is defined:

$a_0=2$; $a_{n+1}=\dfrac{a_n+(5/a_n)}{2}$.

Show that this sequence converges to $\sqrt{5}$, as follows:

(a) Show that $a_n\geq\sqrt{5}$ for $n\geq 1$

The function $f(x)=x+5/x$ has derivative $f'(x)=1-5/x^2$, which vanishes at $x=\pm\sqrt{5}$. Since $f(x)\to\infty$ as $x\to 0^+$ or $x\to\infty$, the unique minimum of $f(x)$, for $x>0$, is at $x=\sqrt{5}$, and $f(x)=2\sqrt{5}$ there. As all $a_n>0$, we find $a_{n+1}\geq\sqrt{5}$ for all $n\geq 0$—as claimed.

(b) Show that the sequence (a_n) is decreasing for $n\geq 1$.

If $a_n\geq\sqrt{5}$ then $5/a_n\leq a_n$, so $a_{n+1}=\dfrac{a_n+(5/a_n)}{2}\leq a_n$.

(c) Deduce that the sequence (a_n) has a limit L.

Apart from the first term, the sequence (a_n) is decreasing and bounded below, so the Dichotomy Theorem applies: there is a limit.

(d) Show that $L=\sqrt{5}$.

$L=\lim a_n=\lim a_{n+1}=\lim\dfrac{a_n+(5/a_n)}{2}=\dfrac{L+(5/L)}{2}$

Solve: $L=\dfrac{L+(5/L)}{2}$, $L=5/L$, $L=\pm\sqrt{5}$. As $L>0$ we have $L=\sqrt{5}$.

Problem **13** is on the next page.

13. p-series with $p = 2$

The mathematician Leonhard Euler showed in 1734 that the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is equal to $\frac{\pi^2}{6}$ (this took him 4 years). Using this fact, calculate the following related sums:

(*a*) $A = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ (odd terms)

(*b*) $B = \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$ (even terms)

(*c*) $C = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ (alternating variant)

Writing S for Euler's sum, we have:

$$S = \frac{\pi^2}{6} \text{ (Euler)}; \quad A = S - B; \quad B = S/4; \quad C = S - 2B$$

.

(Write out the first few terms of each series to see why.)
So $B = \frac{\pi^2}{24}$; then $A = \frac{\pi^2}{8}$, $C = \frac{\pi^2}{12}$.

How could each of these series be used to calculate π?

Take the first few terms and multiply as needed by 6, 8, 24, or 12. Estimates for the error can be obtained using the Integral Test (or in one case, directly from the alternating series). The convergence is not very rapid.

Another method for computing π (John Machin, 1706) would be to use the clever trigonometric identity

$$\frac{\pi}{4} = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$$

and to plug this into the Maclaurin expansion for $\tan^{-1}(x)$, to get a series converging rapidly enough to be of practical value.