Your second midterm examination is likely to contain some problems that do not resemble these review problems.

(19 problems, 4 pages)

Part I. Integration and Differential Equations

(1) Improper Integrals. Evaluate those of the following integrals which converge.

$(a)\ \int_1^\infty \frac{dx}{x+x^3}$ $\quad (b)\ \int_0^\infty \frac{x^2dx}{e^{2x}}$ $\quad (c)\ \int_{-\infty}^\infty \frac{dx}{1+x^2}$ $\quad (d)\ \int_0^{\pi/2} \sec x dx$

$(e)\ \int_{-1}^1 \frac{1}{x^2}dx$ $\quad (f)\ \int_0^1 \ln x dx$ $\quad (g)\ \int_3^\infty \frac{dx}{(x^3-x)^{1/4}}$ $\quad (h)\ \int_0^\infty \frac{\ln(x)}{1+x^2}dx$

(2) Arc length
Find the perimeter of the cardioid $r = 1 - \cos\theta$.

(3) Arc length
Our goal in this problem is to design a test question of the following type: *"Find the length of the graph of the function* $y = cx^2 - \ln x$ *over the interval* $[2, 3]$*."*

We want to choose the constant c so that it is possible to get an exact answer. First find a value of c for which this problem can be solved exactly; and then find the answer.

Hint: For what values of a and b does the equation $1 + (a - b)^2 = (a + b)^2$ hold?

(4) Parametric equations
Find a parametrization $x = f(t)$, $y = g(t)$ of the ellipse $9x^2 + 4y^2 = 36$.

(5) Self-intersections and tangent lines.
(a) The curve given parametrically by $x = t^2 - 9$, $y = t^3 - t$ crosses itself at a point P. Find the coordinates of that point.
(b) Find the equations of the tangent lines at the point P.
(c) Find the angle between the two curves at the point P.

(6) Area
Find the area between the circle $r = 2\cos\theta$ and the cardioid $r = 1 + \cos\theta$.

(7) Surface area
Find the area of the surface obtained by rotating the curve $y = \sqrt{x+1}$, $1 \le x \le 5$ about the x-axis.

MORE ...

(8) Volume
Let R be the region lying within the cardioid $r = 1 - \cos\theta$ and to the right of the y-axis. If a solid S is formed by rotating the region R about the x-axis, write down a formula for the volume of S expressed as an integral. Do not evaluate the integral.

(9) Initial value problems. In each case, find the solution explicitly.
(a) $dx/dt = \tan x$, $x(0) = \pi/6$ (b) $(4 + x^3)^{1/2}(dy/dx) = (xy)^2$, $y(0) = -1$

(10) Heat transfer
A restaurant with an ambient temperature of 25°C serves coffee at the temperature of 50°C. If the coffee cools to 30°C in an hour, and if the customer orders coffee with the main dish but drinks it with dessert at a temperature of 35°C, how long did the customer take to eat the main dish?

(11) Celestial mechanics and mathematical finance
Using Kepler's laws and observational data, an astronomer determines that an asteroid will wipe out all life on earth in exactly 30 years. Armed with this information, she decides to invest all her available funds at 5% interest, and to withdraw $20,000 per year, with the balance declining to 0 when the asteroid hits. How much cash does she have?

(12) Graphical Methods: Slope Fields
Sketch the slope field of $\dot{y} = t^2 - y$ for the region defined by $t \in [-4, 4]$, $y \in [-16, 16]$, and then sketch the graphs of the corresponding initial value problems with $y(0)$ equal to -2, 1, or 2. Discuss the critical points on these three curves.

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Part II. Taylor Polynomials, Sequences, and Series

(13) Taylor Polynomials and Infinite Series

(a) Find the Taylor polynomial of degree 5, $T_5(x)$, for the function $f(x) = \ln(x)$, centered at $x = 1$, and evaluate $T_5(3/2)$.

(b) Estimate the error $|\ln(3/2) - T_5(3/2)|$, using the Error Bound for Taylor Polynomials.

(c) The number $\ln(3/2)$ can be written *exactly* as the alternating sum

$$\ln(3/2) = \frac{(1/2)}{1} - \frac{(1/2)^2}{2} + \frac{(1/2)^3}{3} - \frac{(1/2)^4}{4} + \frac{(1/2)^5}{5} - \frac{(1/2)^6}{6} + \cdots .$$

Use this fact to find another estimate for the error $|\ln(3/2) - T_5(3/2)|$.

(d) Which of these two estimates is the better one?

(14) Limits of sequences Evaluate these limits:

(a) $\displaystyle\lim_{n\to\infty} \left(\frac{\sin(n)}{n}\right)$ (b) $\displaystyle\lim_{n\to\infty} (3n)^{1/n}$ (c) $\displaystyle\lim_{n\to\infty} n(\sin(1/n))$

(d) $\displaystyle\lim_{n\to\infty} n^2(1-\cos(1/n)))$ (e) $\displaystyle\lim_{n\to\infty} \left(1 - \frac{5}{n}\right)^n$

(15) The sum of an infinite Series

Prove that each of the following series converges, and calculate the sum in each case.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{3^n + (-5)^n}{7^n}$, (b) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n(n+1)}$, (c) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n(n+2)}$

(16) Geometric Series

(a) A student applies the formula for the sum of a geometric series as follows: $1 + 2 + 2^2 + 2^3 + 2^4 + \cdots = \dfrac{1}{1-2} = -1$.

What went wrong?

(b) Applying the formula twice more, the same student calculates:

$$\begin{aligned} 1 - 1 + 1 - 1 + 1 \mp \cdots &= \sum_{n=0}^{\infty} (-1)^n = \frac{1}{1-(-1)} = 1/2 \\ .99999\ldots &= .9 \cdot \sum_{n=0}^{\infty} (.1)^n = .9 \cdot \frac{1}{1-(.1)} = 1 \end{aligned}$$

Are these conclusions correct, approximately correct, or simply wrong? Why?

(c) Write the repeating decimal 0.297297297... as a fraction.

(17) The Limit Comparison Test

Use the Limit Comparison Test to determine whether each of the following converges.

$(a)\ \sum_{n=5}^{\infty} \left(1 - \frac{5}{n}\right)^n \quad (b)\ \sum_{n=5}^{\infty} \sin(1/n) \quad (c)\ \sum_{n=5}^{\infty} (1 - \cos(1/n))$

(18) Summation: Error Estimates

Estimate the error involved in taking the sum of the first 10 terms of each of the following series as an approximation to the full sum.

$(a)\ \sum_{n=1}^{\infty} \frac{1}{n^2+n}$

$(b)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}$

(19) Absolute vs. Conditional Convergence

Define *absolute convergence*, and give an example of a series which converges conditionally, but does not converge absolutely.