Math 152 Problems, Week 13

In combinatorics, probability, statistics and theoretical computer science, the idea of a *GENERATING FUNCTION* is very important. Here's the definition:

A function $F(x)$ is the *GENERATING FUNCTION* for a sequence $\{a_n\}_{n=0}^{\infty}$ if

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

1. Find the generating functions for the following sequences.

a) $\{1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{n!}, \ldots\}$

b) $\{1, 1, 1, \ldots$ (all 1's)$\}$

c) $\{1, 3, 3, 1, 0, \ldots$ (all the rest 0's) $\}$

2. Suppose we know the generating function $F(x)$ for some sequence $\{a_n\}_{n=0}^{\infty}$.

a) Show that $xF'(x)$ is the generating function for $\{na_n\}_{n=1}^{\infty}$. (*Hint:* Differentiate the series for $F(x)$ term-by-term and then multiply by x.)

b) Suppose $a_0 = 0$. Show that $\int_0^x \frac{F(t)}{t}\,dt$ is the generating function for $\{\frac{1}{n}a_n\}_{n=1}^{\infty}$. (*Hint:* Integrate the series for $F(t)/t$ term-by-term.)

3. Below is a list of sequences and a list of functions. Match the sequences with their generating functions. Use the Taylor series for the functions that you know and the results of problems #1 and #2.

Sequences :	Functions :
a) $\{0, 1, 2, 3, \ldots, n, \ldots\}$	α) $\dfrac{x + x^2}{(1-x)^3}$
b) $\{1, 2, 4, 8, \ldots, 2^n, \ldots\}$	β) $\ln\left(\dfrac{1}{1-x}\right)$
c) $\{0, 1, 4, 9, \ldots, n^2, \ldots\}$	γ) $\dfrac{1}{1-2x}$
d) $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$	δ) $\int_0^x \ln\left(\dfrac{1}{1-t}\right)\dfrac{dt}{t}$
e) $\{0, 1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\}$	ϵ) $\dfrac{x}{(1-x)^2}$

(OVER)

Remarks on Graphing Curves in Parametric Form: To plot the curve with parametric equations $x = f(t)$ and $y = g(t)$, press MODE key, move cursor down to fourth line, and over to `Par`. Now press $Y =$ key and insert the parametric formulas $X_{1T} = f(T)$ and $Y_{1T} = g(T)$. Then press WINDOW key and set the range for the parameter T and the T_{step} size. (Typically the step size can be .1, but you should experiment to see what happens when you change the step size.) You must also set the $X - Y$ window size as usual. Then press GRAPH. The TRACE command is very useful here, as it displays the T values as well as the $X - Y$ values.

4. Investigate the family of curves defined by the parametric equations $x = t^2$, $y = t^3 - ct$, where $c > 0$ is a constant.

(a) Graph the curves for $c = 1/4$, $c = 1$ and $c = 4$. What features do all the curves have in common? (You may need to adjust the graphing window as you change c.) How does the shape change as c increases?

(b) How many tangent lines does the curve have at the point $(c, 0)$? What are their slopes? Use the parametric formula

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}, \qquad \dot{x} = \frac{dx}{dt}, \quad \dot{y} = \frac{dy}{dt}$$

Check your answer numerically (for $c = 1/4$, $c = 1$ and $c = 4$) by drawing the tangent lines on your graphing calculator. Press `TRACE`, move the cursor to the desired spot on the curve, press the `DRAW` key, select `Tangent(` and then press `ENTER`. You can also use the `CALC` key and select dy/dx (press `ENTER` twice to get the numerical value for the slope dispayed on the screen.

5. A point P travels on the parabola $y = x^2$.

(a) Give parametric formulas for the location of P with parameter the angle θ that the ray from the origin to P makes with the positive x-axis. Draw a picture. The parametric formulas will have to be given "piecewise".

(b) Give parametric formulas for the location of P with parameter the distance r from the origin to P. Draw a picture. The parametric formulas again will have to be given "piecewise".