Intro to Mathematical Reasoning (Math 300)

Supplement 7. Solutions to problems from supplement 6. [1]

Problem 6.1 Prove: For all integers a, b, c, if a is a divisor of b and b is a divisor of c then a is a divisor of c.

Proof. Let a, b, c be integers such that a is a divisor of b and b is a divisor of c. We want to show a is a divisor of c which means we want to show that there is an integer k such that $ak = c$.

Since a is a divisor of b there is an integer which we call w such that $wa = b$.

Since b is a divisor of c there is an integer which we call v, such that $vb = c$.

Therefore $c = vb = vwa$. Since v and w are integers vw is also. Therefore a is a divisor of c.

□

A comment on the above proof: In the problem statement a, b, c are dummy variables occurring with a universal quantifier. We use the arbitrary value method to prove this. In previous supplements, when we introduced objects to stand for a, b, c we used different letters for the object names than a, b, c. This is not necessary, and was done to emphasize the difference between object names and dummy variables. Hopefully you are now getting used to this difference and so we now use a, b, c for the object names, which is simpler and clearer. Notice that doing this this does not violate the guidelines for using letters. A letter that is used as a dummy variable may be used later as an object name. (However, you are not allowed to do the reverse; an active object name may not be used as a dummy variable.)

Problem 6.2 Prove: No integer is both even and odd. (*Comment: Here you should use the following definition of odd: an integer n is odd provided that there is an integer j such that $n = 2j + 1$.*)

Proof. The statement to be proved is equivalent to the universal statement: For all integers n, it is not the case that n is even and n is odd.

Let n be an arbitrary integer. We need to prove that it is not the case that n is even and n is odd. We will use proof by contradiction. Assume, for contradiction, that n is even and n is odd. Since n is even, there is an integer which we will call j such that $2j = n$. Since n is odd, there is an integer which we will call k such that $2k + 1 = n$.

Then $2j = 2k + 1$ which implies $2(j - k) = 1$. The number $j - k$ is the difference of two integers and so is an integer. If $j - k$ was greater than or equal to 1 then $2(j - k) \geq 2$ which is impossible and if $j - k \leq 0$ then $2(j - k) \leq 0$ which is also impossible. Then $0 < j - k < 1$ which contradicts that $j - k$ is an integer. We therefore conclude that n is not both even and odd.

□.

Problem 6.3 Each integer is even or odd.

Proof. We want to prove: For all integers n, n is even or n is odd.

[1] Version 2/16/04.

We will prove separately: (1) for all nonnegative integers n, n is even or n is odd, and (2) for all negative integers n, n is even or n is odd.

To prove (1), we use the principle of mathematical induction. Let n be a nonnegative integer. The base case is the case $n = 0$. Then n is even since $n = 2 \times 0$.

The induction case is the case $n \geq 1$. By the induction hypothesis, $n - 1$ is even or odd. We divide into two cases depending on whether $n - 1$ is even or not.

Subcase 1. Assume $n - 1$ is even. So there is an integer we will call j such that $n - 1 = 2j$. Then $n = 2j + 1$, so n is odd.

Subcase 2. Assume $n - 1$ is not even. Then by the induction hypothesis, since $n - 1$ is even or odd, $n - 1$ must be odd. So there is an integer we will call j such that $n - 1 = 2j + 1$. Then $n = 2j + 2 = 2(j + 1)$, so n is even.

In either subcase we conclude n is even or odd, which completes the proof of the induction case. Therefore (1) is proved.

Next we prove (2), that for all negative integers n, n is even or n is odd. Let n be a negative integer. Let $m = -n$. Then m is a positive integer so, by (1), which we have already proved, m is even or odd. We consider two cases, depending on whether m is even or odd.

Case 1. Assume m is even. Then there is an integer we'll call j such that $m = 2j$. Then $n = -m = -2j = 2 \times (-j)$ and so n is also even.

Case 2. Assume m is odd . Then there is an integer we'll call j such that $m = 2j + 1$. Then $n = -m = -2j - 1 = 2(-j - 1) + 1$ and so n is odd.

This completes the proof of (2). Since we have shown (1) and (2) are true, this completes the proof of the statement we originally set out to prove. □

Problem 6.4 Every rational number can be expressed as the quotient of two integers that have no common factor divisor greater than 1.

Proof. First we prove: every a rational number can be expressed as the quotient of an integer and a positive integer. Let r be rational. Then by definition there are integers that we'll call a, b with $b \neq 0$ such that $r = a/b$. If $b > 0$ then we have r expressed as a quotient of an integer a and a positive integer b. Otherwise, $b < 0$ and so $-b$ is a positive integer and r is equal to the quotient of $-a$ and $-b$.

Now, since every rational can be expressed as the quotient of an integer and a positive integer, the statement we want to prove is equivalent to the following:

> Theorem. For all positive integers n and integers m, there exist integers r and s that have no common divisor greater than 1 and such that $m/n = r/s$.

To prove this let n be a positive integer and m be an integer. We must prove that there exist integers r and s that have no common divisor greater than 1 and such that $m/n = r/s$ such that $m/n = r/s$.

We prove this by strong induction on n. For the base case, assume $n = 1$. Then m and n have no common divisor greater than 1, so we may take r to be m and s to be n and the conclusion of the theorem is satisfied.

For the induction case, assume $n > 1$. By the induction hypothesis, we may assume that for all positive integers k that are less than n and for all integers j, that there exist integers r and s that have no common divisor greater than 1 and such that $j/k = r/s$. We divide into two cases depending on whether or not m and n have a common divisor greater than 1 or not.

Subcase 1. Assume m and n have no common divisor greater than 1. Then we may take $r = m$ and $s = n$ in the conclusion of the theorem.

Subcase 2. Assume m and n have a common divisor greater than 1. Let d be a common divisor of n and m that is greater than 1. Let $n' = n/d$ and $m' = m/d$ so that $m'/n' = m/n$, Now n' is a positive integer less than n, m' is an integer so by the induction hypothesis there are integers, which we will call u and v, that have no common divisor greater than 1 such that $m'/n' - u/v$. Then also $m/n = u/v$ to complete this case.

In either subcase we obtain the required conclusion and therefore the induction case is completed, to complete the proof. □

Problem 6.5 There is no rational number whose square is equal to 2.

Proof. We prove this by contradiction. Suppose for contradiction there is a rational number whose square is equal to 2. Let r be such a number. By the result of problem 6.4, there exist integers which we will call a and b such that $r = a/b$, and a and b have no common divisor greater than 1. Since $r^2 = 2$ we have $a^2 = 2b^2$. We consider two cases, depending on whether a is odd or a is even.

Case 1. Assume a is odd. Since the product of odd numbers is odd, a^2 is also odd. But since b is an integer then b^2 is also an integer so $a^2 = 2b^2$ is even. Then a^2 is both even and odd, which, by problem 6.1, is impossible.

Case 2. Assume a is even. So there exists an integer which we will call j such that $2 = 2j$. Then $a^2 = (2j)^2 = 4j^2$. This implies $4j^2 = 2b^2$, which implies $2j^2 = b^2$.

Since b has no common divisor with a, b is not even, so by problem 6.2, b is odd, which implies b^2 is odd. But $b^2 = 2j^2$ and j^2 an integer imply that b^2 is even, which is impossible by problem 6.1.

Since both cases lead to a contradiction, we conclude that there is no rational number whose square is 2.

□.

Problem 6.6 Let U be a set and let $\mathcal{A}$ be a set of subsets of U having the property that for all $A, B \in \mathcal{A}$, $A \cap B \neq \emptyset$. Then for all subsets Y of U, one of the following two conclusions hold:

(1) For all $A \in \mathcal{A}$, $A \cap Y \neq \emptyset$, or (2) For all $A \in \mathcal{A}$, $A \cap Y^c \neq \emptyset$. (*Comment: Notice that in the statement of this theorem* U*, and* $\mathcal{A}$ *are introduced by "Let" statements. This means that the conclusion of the theorem should be true for any choice of* U *and* $\mathcal{A}$ *satisfying the hypotheses. Since* U *and* $\mathcal{A}$ *are introduced in the theorem statement, they are object names that can be referred to in the proof without further introduction. On the other hand,* A, B *and* Y *are dummy variables that can not be treated as object names.*)

Proof. Let Y be an arbitrary subset of U. We must show that for all A in $\mathcal{A}$, $A \cap Y \neq \emptyset$ or for all $A \in \mathcal{A}$, $A \cap Y^c \neq \emptyset$.

Since the conclusion we are trying to prove is an "or" of two statements, we are allowed to assume that the first of these statements if false. We must then show that the second must be true. [*Comment: This is one of the allowed proof strategies for the "or" of two statements discussed in handout 5.*]

So assume that it is not the case that for all $A \in \mathcal{A}$, $A \cap Y \neq \emptyset$.

Then there is a set belonging to $\mathcal{A}$, which we will call Z, such that $Z \cap Y = \emptyset$. Then for all $z \in Z$, $z \notin Y$, which means that $z \in Y^c$. Thus $Z \subseteq Y^c$.

We now show that for all $A \in \mathcal{A}$, $A \cap Y^c \neq \emptyset$. Let W be an arbitrary member of $\mathcal{A}$. We must show that $W \cap Y^c \neq \emptyset$. Now, by the hypothesis on $\mathcal{A}$, $W \cap Z \neq \emptyset$, which means that there is an element, which we will call z such that $z \in W$ and $z \in Z$. Since $Z \subseteq Y^c$, we have $z \in Y^c$. Therefore $z \in W$ and $z \in Y^c$ and thus $z \in W \cap Y^c$ and thus $W \cap Y^c \neq \emptyset$. □

Problem 6.7. Prove that for every positive real number R there is a positive real number z such that for all x belonging to the interval $(1-z, 1)$, $1/(1-x) \geq R$.

We will give two proofs for this statement and compare them afterwards.

Proof 1. Let R be a positive real number. We must show that there is a positive real number z such that for all x belonging to the interval $(1-z, 1)$, $1/(1-x) \geq R$.

To find such a z, we work backwards from the conclusion and ask the question: for which numbers x that are less than 1 is it the case that $1/(1-x) \geq R$? Let x be a real number less than 1. Then $(1-x)/R$ is positive and so $1/(1-x) \geq R$ is equivalent to the condition obtained by multiplying both sides by $(1-x)/R$. Therefore $1/(1-x) \geq R$ if and only if $1/R \geq 1-x$. Adding $x - 1/R$ to both sides of this new inequality, we obtain that $1/(1-x) \geq R$ if and only if $x \geq 1 - 1/R$. Since $x < 1$ we deduce that if $x < 1$ and $x \geq 1 - 1/R$ then $1/(1-x) \geq R$. Therefore if x belongs to the interval $(1-1/R, 1)$ then $1/(1-x) \geq R$. Thus if we take $z = 1/R$ we have that every x belonging to $(1-z, 1)$ satisfies $1/(1-x) \geq R$. □

Proof 2. Let R be a positive real number. We must show that there is a positive real number z such that for all x belonging to the interval $(1-z, 1)$, $1/(1-x) \geq R$.

Let $z = 1/R$. Then z is positive since R is. We claim that for all x belonging to the interval $(1-z, 1)$, $1/(1-x) \geq R$. Let x be a number belonging to $(1-z, 1)$ which means $1 - z < x$ and $x < 1$. Since $1 - z < x$ we can add $z - x$ to both sides to deduce $1 - x < z$. Since $1 - x$ and z are both positive, we may multiply this inequality by $1/z(1-x)$ to deduce $1/z < 1/(1-x)$. Since $1/z = R$ we conclude that $R < 1/(1-x)$ as required. □

How do these proofs compare? The two proofs start the same way and use the standard proof strategy for universal-existential statements. In both proofs we need to find a z that satisfies certain requirements. In the first proof, we work backwards from the conclusion to find the z. So the value we take for z is not stated until the end.

In the second proof, we start by stating the choice of z and then show that it works, so we work forwards towards the conclusion.

The advantage of the first proof is that it explains to the reader how z was determined. In the second proof, we choose z to be $1/R$ but this is very mysterious. The reader is left to wonder: Where did that come from. Why was z chosen this way? (The answer is: the writer of the proof did some "scratch work" to find z as in the first proof but did not include the scratch work in the proof.)

The advantage of the second proof is that the logic of the proof is clearer since we are working forwards to the conclusion instead of working backwards.

The choice of which proof is better is a matter of taste. The author of these notes prefers the first proof (which explains the reason for the choice of z), but many mathematicians prefer the second proof.