Intro to Mathematical Reasoning (Math 300)
Supplement 8. Terminology related to functions and relations [1]

This supplement has some corrections and clarifications to the terminology for relations and functions in chapters 4 and 5 of the text.

Before getting into the details, let's begin with some general remarks about definitions. First, let's recall the purpose of making definitions in mathematics: to allow ideas to be expressed more concisely and clearly. For example, when studying the integers, we notice that some positive integers have no positive divisors other than themselves or 1. Such numbers are important, and we want to talk about them. It is tedious to keep saying things like "Let n be a positive integer that has no positive divisors other than itself or 1." So we make a definition: We define the word "prime" in the context of integers by saying that: A positive integer is said to be prime if it has no positive divisors other than itself and 1. Once we have made such a definition, we are free to use it to shorten the above introduction of n to "Let n be a prime."

When we make a mathematical definition, we are assigning a specific meaning to a word or symbol. In your own mathematical writing, you are free to make any definitions that will make it easier to express your ideas clearly.

Certain mathematical concepts are so important and come up so frequently that there is general agreement among mathematicians about certain definitions relating to those concepts. The term "prime" above is an example.

In many cases, there is a minor disagreement among mathematicians about what a term should mean. For example, some authors use the word "natural number" to mean a positive integer, while others use it to mean a nonnegative integer. The difference is that under the first definition, 0 is not a natural number and under the second, it is a natural number.

There are similar minor disagreements with some of the terminology associated to relations and functions. The author of our textbook has made certain choices about these definitions which have some disadvantages. There are also some minor inconsistencies in the author's choices. In this supplement, we'll revise and clarify some of these basic definitions.

Important Note: Instructors may have differing ideas about the preferred way to define the terminology of relations and functions. Not all instructors will agree with the recommendations given here. Before reading this supplement, you should check with your instructor about whether he or she is using the definitions given here, or is using an alternative set of definitions.

1 Definitions and facts about relations

In definition 4.1.4, the book defines two notions:

- The notion of a *relation between two sets A and B*
- The notion of a *relation on a set A*

However, the book never defines the word *relation* on its own. Here is the standard definition:

A *relation* is a set all of whose members are ordered pairs.

(Notice that the empty set is a relation.) With this definition we can restate definition 4.1.4 as:

Let A, B be sets and R be a relation.

- If $R \subseteq A \times B$ then we say R is a relation *from A to B*. (Note: the book calls this a relation *between A and B*.)
- If $R \subseteq A \times A$ then we say R is a relation on A.

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It is important to realize that the same relation R can be viewed as a relation from sets A to B for different pairs A and B.

Example. Consider the relation $R = \{(1, 11), (2, 12), (3, 12)\}$. Then R is a relation from the sets $\{1, 2, 3\}$ to $\{11, 12\}$. It is also correct to say that R is a relation from $\{1, 2, 3, 4, 5\}$ to $\{8, 11, 12, 14\}$ and that R is a relation from $\mathbb{N}$ to $\mathbb{N}$. R is a relation on the set $\{1, 2, 3, 11, 12\}$ and is also a relation on $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ and on $\mathbb{R}$.

Example. Consider the relation $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. C includes the points $(1, 0)$, $(0, -1)$, and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. C is a relation on $\mathbb{R}$. It is also correct to say that C is a relation on the set $[-1, 1]$ (the closed interval from -1 to 1) since $C \subseteq [-1, 1] \times [-1, 1]$.

The book defines the notion of domain and image of a function but it also makes sense to define the domain and image of any relation:

- For a relation R, the *domain* of R, denoted $Dom(R)$ is the set of all first elements of pairs in R. More precisely, $Dom(R) = \{a : \exists x \text{ such that } (a, x) \in R\}$.
- For a relation R, the the *range* of R or the *image* of R, denoted $Ran(R)$ is the set of all second elements of pairs in R. More precisely, the $Ran(R) = \{a : \exists x \text{ such that } (x, a) \in R\}$.

In the first example above, $Dom(R) = \{1, 2, 3\}$ and $Ran(R) = \{11, 12\}$. In the second example above, $Dom(C) = Ran(C) = [-1, 1]$.

Proposition. Let A, B be sets and let R be a relation. Then R is a relation from A to B if and only if $Dom(R) \subseteq A$ and $Ran(R) \subseteq B$. R is a relation on A if and only if $Dom(R) \cup Ran(R) \subseteq A$.

In definition 4.1.8, the book introduces the term *reflexive* which applies to a relation R defined on a set A. The set A is an important part of the definition.

For example, consider the relation $Q = \{(i, j) \in \mathbb{Z}^2 : i \leq j\}$. Q can be viewed as a relation on the set $\mathbb{Z}$. But it can also be viewed as a relation on the set $\mathbb{R}$. As a relation on $\mathbb{Z}$, Q is reflexive, but as a relation on $\mathbb{R}$, Q is not reflexive.

The book does not define the inverse of a relation, but we will:

> If R is a relation, the *inverse of* R is the relation obtained by switching the order of each pair in R. More precisely, the inverse of R is the set $\{(x, y) : (y, x) \in R\}$.

The inverse of the relation R in the first example above is $\{(11, 1), (12, 2), (12, 3)\}$. The inverse of the relation C in the second example is C itself.

Proposition. Let R be a relation and Q be its inverse. Then $Dom(Q) = Ran(R)$ and $Ran(Q) = Dom(R)$.

2 Definitions and facts about functions

In definition 5.1.1, the book defines the notion of a function from a set A to a set B. However, the book never defines the term *function* on its own. Here is a definition:

> A function is a relation f satisfying the property that for any two pairs $(a, b), (d, c)$ in f, if $a = d$ then $b = c$.

Informally, a relation f is a function provided that the first element of any two different pairs belonging to f are different.

Since a function is a relation, it is a set of ordered pairs. You should check that the empty set satisfies the requirements of a function.

Since a function is a relation, and we have defined the domain and range of an arbitrary relation, these definitions also apply to an arbitrary function. These definitions agree with the definition 5.1.5 in the book.

Next we define the notation $f(x)$:

> Let f be a function. For $x \in Dom(f)$, there is one and only one ordered pair in f having x as its first element. We define $f(x)$ to be the unique element such that $(x, f(x)) \in f$. For $x \notin Dom(f)$ we say that $f(x)$ is *undefined.*

Here is a definition that is not in the book. A *target set* for a function f is any set that contains $Ran(f)$. It is important to keep in mind that a function has many possible target sets.

Definition 5.1.1 gives the definition of what it means that f is a function from A to B. Here is a restatement of this definition using the notion of a target set.

> If A and B are sets and f is a function we say that f is a function from A to B, written $f : A \longrightarrow B$ if A is the domain of f and B is a target set for f.

Example. Consider the relation $\{(1,0),(2,1),(3,0),(4,2)\}$. This is a function because the first element of different pairs are different. The domain is $\{1,2,3,4\}$ and the range is $\{0,1,2\}$. Any set containing $\{0,1,2\}$ is a target set for f, for example, $\mathbb{N}$ is a target set. We can say that f is a function from $\{1,2,3,4\}$ to $\{0,1,2\}$ and also that f is a function from $\{1,2,3,4\}$ to $\{0,1,2,10,11,100\}$.

We sometimes introduce a function by writing something like "Let $f : A \longrightarrow B$ be a function." The meaning of this sentence depends on whether A and B have been previously introduced or not.

- If A and B are names for sets that are already introduced, then this sentence is a shortcut for the following sentence: "Let f be a function whose domain is A and whose range is contained in B".
- If A and B have not been previously introduced, then this is a shortcut for "Let A and B be sets. Let f be a function whose domain is A and whose range is contained in B."
- If A was previously introduced but B wasn't then this is a shortcut for "Let B be a set. Let f be a function whose domain is A and whose range is contained in B."
- If B was previously introduced but A wasn't then this is a shortcut for "Let A be a set. Let f be a function whose domain is A and whose range is contained in B."

Warning. In definition 5.1.5, the book makes the definition of *the codomain of a function* f. The author says that if f is a function from A to B then B is the codomain of f and is denoted $Codom(f)$. But this does not make sense because for the function f, there are many possible target sets, so $Codom(f)$ does not have a clear meaning. *Therefore, the author of this supplement recommends that you not use the notation* $Codom(f)$.

On page 106, the author says "Two functions are equal if they have the same domain, the same codomain and agree on every element of the domain." This is not quite correct, we will now correct this.

Since a function is, by definition, a set of ordered pairs (satisfying a particular condition) when we say that two functions f and g are equal we mean that they satisfy the requirements of equality for two sets given by definition 2.2.7: $f \subseteq g$ and $g \subseteq f$, or, in other words, every ordered pair belonging to f must belong to g and every ordered pair belonging to g must belong to f. The following theorem, which is similar to Theorem 5.1.7 gives another criterion for telling that two functions are equal.

Theorem. Let f and g be functions. Then $f = g$ if and only if $Dom(f) = Dom(g)$ and for all $x \in Dom(f)$, $f(x) = g(x)$.

Specifying a function. If we want to specify a function f, we can specify it as a set of ordered pairs. For example we might say "Let f be the function $\{(1, 2), (2, 3), (3, 4)\}$" or "Let g be the function $\{(n, 2n) : n \in \mathbb{Z}\}$". This second introduction means that g contains all pairs obtained by substituting an integer for n in the ordered pair $(n, 2n)$. There is another way a function, which is the one you are most familiar with: A function can be specified by (1) specifying the domain, and (2) specifying a rule which, given an arbitrary element a of the domain specifies how to determine $g(a)$. For example, we could specify the function g above by saying "Let g be the function with domain $\mathbb{Z}$ given by the rule $g(n) = 2n$ for all $n \in \mathbb{Z}$."

The terms *one-to-one*, *onto* and *one-to-one correspondence* introduced in the book on page 106 need to be used carefully. Here is a careful definition of the three concepts (replacing definition 5.1.8):

Definition. Let f be a function.

- f is one-to-one if for each b in $Ran(f)$ there is exactly one $a \in A$ for which $b = f(a)$.
- If B is a target set for f, we say that f *maps onto* B or that f *is onto for the target set* B if $Ran(f) = B$. In other words, f maps onto B if for each $b \in B$ there is an $a \in Dom(f)$ such that $f(a) = b$.
- If B is a target set for f, we say that f *is a one-to-one correspondence onto* B if f is one-to-one and maps onto B.

Some remarks:

1. The terms "one-to-one" and "one-to-one correspondence" do not mean the same thing. Be careful not to confuse them.
2. For any function f there is a set B such that f maps onto B, namely $B = Ran(f)$.
3. Strictly speaking, the statements "function f is onto" or "function f is a one-to-one correspondence" do not make sense because the word "onto" and the phrase "one-to-one correspondence" only make sense relative to a specific target set. However, if f is introduced as a function from set A to set B, and later say that "f is onto" we mean: "f is onto for the target set B" and if we later say "f is a one-to-one correspondence" we mean that f *is a one-to-one correspondence onto* B.
4. The following is a minor change in the second gray box on page 108: If f is a function and B is a target set for f, then the statement that f maps onto B is a "universal-existential statement": for all $b \in B$ there exists $a \in Dom(f)$ such that $f(a) = b$. To prove this we use the standard proof strategy for universal-existential statements: Let b be an arbitrary member of B and then give instructions for producing a such that $a \in Dom(f)$ and $f(a) = b$.

Theorem 5.1.16 is incorrect as stated. The point that the author is trying to make in that theorem is the point made in the second remark above.

Here is a reformulation of the definition of composition (5.2.1):

> Let f and g be functions. If $Ran(f) \subseteq Dom(g)$ then we define the function $g \circ f$, which has domain $Dom(f)$ and is given by the rule $g \circ f(a) = g(f(a))$ for all $a \in Dom(f)$.

If $Ran(f)$ is not a subset of $Dom(g)$ then $g \circ f$ has no meaning we say $g \circ f$ is *not well defined*.

It follows from the above definition that if A,B and C are sets and f is a function from A to B and g is a function from B to C then $g \circ f$ is a function from A to C. This is what is stated in definition 5.2.1.

Finally, let us clarify the discussion of inverse functions beginning on page 111. In the previous section of this supplement, we defined the inverse of an arbitrary relation R to be the set of pairs $\{(x, y) : (y, x) \in R\}$. Since a function is a relation, the inverse of f is defined in the same way. We can write the inverse of f as the following set: $\{(f(a), a) : a \in Dom(f)\}$. The inverse of f might be a function or it might not be. The following theorem replaces Theorem 5.2.7:

Theorem. Let f be a function. The inverse of f is a function if and only if f is one-to-one. Furthermore, the domain of the inverse of f is equal to the range of f.

The inverse of f is written f^{-1}. The book only uses the notation f^{-1} if f^{-1} is a function.