Mathematics 373 Workshop 2 Solutions

Iteration

Fall 2003

Problem 1

Let

$$g(x) = \frac{1}{x^2} - 3$$

Let $x_0 = -3$ and define x_n for $n = 1, 2, 3, \ldots$ by

$$x_n = g(x_{n-1})$$

Such calculations are easily done on a programmable calculator, since the calculation of $g(x)$ can be assigned to a single key on the calculator.

1a Statement

Obtain the values of x_1, x_2, x_3, x_4, x_5. Is there a pattern to the successive values?

If the sequence of values obtained in this way converges (and if g is a continuous function — as all functions in this course will be), then the limit x_∞ satisfies the **fixed point** condition $g(x_\infty) = x_\infty$.

1a Solution

Here are the values

$$x_1 = -2.888888889$$
$$x_2 = -2.880177515$$
$$x_3 = -2.879451589$$
$$x_4 = -2.879390800$$
$$x_5 = -2.879385707$$

It appears that there is a new decimal digit every one or two steps that remains in all later steps.

1b Statement

Do you think there is a limit? If so, what is it (to 10 decimal places)? If not, what indication do you see of failure to converge?

(The explanation of the behavior of iteration is given by the Mean Value Theorem:

$$\frac{g(x) - g(y)}{x - y} = g'(\xi)$$

where ξ is between x and y. If $|g'(x)| < 1$ on an interval, application of g brings points closer together. In particular, if y is one of the fixed points of g, the sequence $x_0, x_1, x_2, \ldots$ converges to y. On the other hand, If $|g'(x)| > 1$ on an interval containing a fixed point, iteration pushes points away from the fixed point.)

1b Solution

This decreasing size of difference of consecutive terms is the sign of a **convergent** sequence. Continuing the iteration, we find $x_9 = -2.8793852416$ with no change after that.

1c Statement Find $\mathcal{G} = \left\{ x : \left|g'(x)\right| < 1 \right\}$. You should get two intervals. One of these intervals contains a fixed point. Iterating g on this interval gives a sequence converging to the fixed point. How does the sequence x_n behave if you start in the other interval in $\mathcal{G}$?

1c Solution Differentiating gives $g'(x) = -2x^{-3}$. To solve the inequalities to find where this is less than 1 in absolute value, it is necessary to split into cases depending on the sign of x. The sign of $g'(x)$ is always opposite to that of x. If $x > 0$, then we need to solve $-2x^{-3} > -1$, which leads to $2x^{-3} < 1$, $2 < x^3$, $x > 2^{1/3} \approx 1.26$. If $x < 0$, then we need to solve $-2x^{-3} < 1$, which leads to $-2 > x^3$, $x < -2^{1/3} \approx -1.26$. Thus

$$\mathcal{G} = \left(-\infty, -2^{1/3}\right) \cup \left(2^{1/3}, \infty\right).$$

There is a fixed point in the interval of all $x < -1.26$ and $0 < g'(x) < 1$ on this interval. The **mean value theorem** shows that $g(x)$ is on the same side of the fixed point as x and closer to the fixed point. In particular, **any** closed interval including the fixed contained in this interval will satisfy the hypotheses of the **fixed point theorem**.

For all x, $g(x) = g(-x)$, so the image of an $x > 1.26$ is in the interval where $x < -1.26$. By our analysis of this interval, continued iteration of g gives points that remain in this interval.

There are two other roots of $x = g(x)$ — at $x = -.652703644666$ and $x = 0.532088886238$, but these are **repelling** fixed points. A different method is needed to find them

Problem 2

Let $h(x) = \cos 2x$. The result of Problem 1 of workshop 1 may be interpreted as saying the $h(x)$ has a unique fixed point.

2a Statement Find 10 terms of the sequence obtained by iterating h starting from $x_0 = 0$. Does it look like this sequence will converge?

2a Solution Here are the values

$$\begin{aligned}
x_0 &= 0. \\
x_1 &= 1. \\
x_2 &= -.416146836547 \\
x_3 &= 0.673181412986 \\
x_4 &= 0.222554106304 \\
x_5 &= 0.902564079142 \\
x_6 &= -.232193130121 \\
x_7 &= 0.894096612150 \\
x_8 &= -.215688525488 \\
x_9 &= 0.908390835689 \\
x_{10} &= -.243512126444
\end{aligned}$$

This is **very different** from what we saw in Problem 1. The values alternate between negative values and values close to $+1$, sometimes by way of smaller positive values.. This does not look like convergent behavior.

2b Statement To test whether this fixed point can be found by iterating h, find an interval I where $|h'(x)| < 1$. I is an open interval. The endpoints of I are points where $|h'(x)| = 1$ (since $h'(x)$ is continuous).

2b Solution Calculus gives $h'(x) = -2\sin(2x)$. By the results of Workshop 1, it suffices to consider $x \in [-1, 1]$, so we can solve this as follows.

$$\begin{aligned} -1 < -2\sin(2x) &< 1 \\ \frac{1}{2} > \sin(2x) &> -\frac{1}{2} \\ \frac{\pi}{6} > 2x &> -\frac{\pi}{6} \\ \frac{\pi}{12} > x &> -\frac{\pi}{12} \end{aligned}$$

Here, $\pi/12 \approx .261799387799$.

2c Statement Add the endpoints to I to obtain a closed interval $\bar{I}$, and determine the maximum and minimum of $h(x)$ on $\bar{I}$. These are the endpoints of interval of values $h(x)$ for $x \in I$.

2c Solution On the interval $\bar{I} = [-\pi/12, \pi/12]$, $h(x)$ increases from $\sqrt{3}/2 \approx .866025403785$ to 1 and then decreases to $\sqrt{3}/2$. These values are all **outside** $[-\pi/12, \pi/12]$.

2d Statement $h(x)$ is a decreasing function for $0 \leq x \leq \pi/2$. Find an inverse function of the restriction of h to this interval. Show that the interval $[0, 1]$ is taken into itself by this function.

2d Solution An inverse function to $h(x)$ is found by solving the equation $y = h(x)$ for x in terms of y. The restriction $0 \leq x \leq \pi/2$ allows this to be done in terms of the arccos function, as follows.

$$\begin{aligned} y &= \cos 2x \\ \arccos y &= 2x \quad (\text{since } 0 \leq 2x \leq \pi) \\ x &= \frac{1}{2}\arccos y \end{aligned}$$

If $0 \leq y \leq 1$, then $\pi/2 \geq \arccos y \geq 0$ and $\pi/4 \geq (1/2)\arccos y \geq 0$. Since $\pi/4 \approx 0.7854$, the interval $[0, 1]$ is taken into itself by this mapping.

2e Statement Find a suitable interval J such that iterating this function starting from $x_0 \in J$ will always converge. Perform such an iteration and compare results to your previous solution of $x = \cos 2x$ by bisection.

2e Solution A direct expression for the derivative of $j(y) = (1/2)\arccos y$ is

$$j'(y) = \frac{-1}{2\sqrt{1 - y^2}}.$$

In the interval $[0, 1]$, this is between -1 and 0 for $0 \leq y \leq \sqrt{3}/2 \approx 0.866$, with the value -1 attained only at the right endpoint. Since $j(0) = \pi/4 < \sqrt{3}/2$, the left endpoint of J can be taken to be zero and the right endpoint can be any value greater than or equal to $\pi/4$ and **strictly** less than $\sqrt{3}/2$. Many **smaller** intervals will also work.

Since $j'(y) < 0$ throughout this interval, the sequence formed by iterating j will oscillate around the fixed point. The slowest convergence results from a starting value of $y_0 = 0$. This is illustrated below. Any other starting point will converge faster.

$$
\begin{aligned}
y_1 &= 0.7853981635\\
y_2 &= 0.3337286080\\
y_3 &= 0.6152700670\\
y_4 &= 0.4540339095\\
y_5 &= 0.5497543540\\
y_6 &= 0.4943630946\\
y_7 &= 0.5268471750\\
y_8 &= 0.5079547125\\
y_9 &= 0.5189938120\\
y_{10} &= 0.5125614655\\
y_{11} &= 0.5163155270\\
y_{12} &= 0.5141266360\\
y_{13} &= 0.5154036170\\
y_{14} &= 0.5146588750\\
y_{15} &= 0.5150932935\\
y_{16} &= 0.5148399190\\
y_{17} &= 0.5149877090\\
y_{18} &= 0.5149015080\\
y_{19} &= 0.5149517875\\
y_{20} &= 0.5149224605\\
y_{21} &= 0.5149395665\\
y_{22} &= 0.5149295890\\
y_{23} &= 0.5149354085\\
y_{24} &= 0.5149320140\\
y_{25} &= 0.5149339940\\
y_{26} &= 0.5149328395\\
y_{27} &= 0.5149335125\\
y_{28} &= 0.5149331200\\
y_{29} &= 0.5149333490\\
y_{30} &= 0.5149332155\\
y_{31} &= 0.5149332935\\
y_{32} &= 0.5149332480\\
y_{33} &= 0.5149332745
\end{aligned}
$$

$$y_{34} = 0.5149332590$$
$$y_{35} = 0.5149332680$$
$$y_{36} = 0.5149332625$$
$$y_{37} = 0.5149332660$$
$$y_{38} = 0.5149332640$$
$$y_{39} = 0.5149332650$$
$$y_{40} = 0.5149332645$$
$$y_{41} = 0.5149332650$$
$$y_{42} = 0.5149332645$$

After this, the evaluation of the function is not accurate enough to show any improvement. Note that $[y_{26}, y_{27}]$ is inside the interval [0.5149327, 0.5149337] at which we stopped our bisection calculation. This is also the first time that the absolute value of a difference between consecutive y_i is less than 10^{-6}. Since the terms oscillate around the fixed point, this difference is an upper bound on the distance to the fixed point. To get an equivalent result when iterating an increasing function, **two** iterations could be used: one starting from a value smaller than the fixed point and one from a value larger than the fixed point.