Mathematics 373 Workshop 4 Solutions

Interpolation Formulas

Fall 2003

Problem 1 Recall the polynomial

$$g(x) = x^4 - 172x^3 + 11084x^2 - 317169x + 3400321.$$

from Problem 2 of Workshop 1.

1a Statement Since $g(x)$ has integer coefficients, you can find its exact value at integer values of x. Begin by finding the exact values of $g(41)$, $g(42)$, $g(43)$, $g(44)$, $g(45)$ (if you suspect that your calculator is not able to give these exact values, use Maple, a more powerful calculator, or calculator-assisted hand computation).

1a Solution The values are

x	$g(x)$
41	-55
42	-41
43	-33
44	-43
45	-59

1b Statement Write the Lagrange interpolation formula for the polynomial of degree four with the values you found in (a). Simplify the coefficients (they will be fractions with denominator at most 24), but **do not expand the polynomials** that multiply your values of $g(x_i)$. (Theorem 3.2 says that this is the same polynomial as g, but we are only interested in computing with this expression, not in verifying the theorem algebraically.)

1b Solution The polynomial is

$$\begin{aligned}
&\left(\frac{-55}{24}\right)(x-42)(x-43)(x-44)(x-45)\\
&\quad+\left(\frac{-41}{-6}\right)(x-41)(x-43)(x-44)(x-45)\\
&\quad\quad+\left(\frac{-33}{4}\right)(x-41)(x-42)(x-44)(x-45)\\
&\quad\quad\quad+\left(\frac{-43}{-6}\right)(x-41)(x-42)(x-43)(x-45)\\
&\quad\quad\quad\quad+\left(\frac{-59}{24}\right)(x-41)(x-42)(x-43)(x-44)
\end{aligned}$$

1c Statement Let $a = 42.90732471$ — a number of no particular significance — and compare the results of calculating $g(a)$ in two different ways: (i) adding the terms in the original expression for g; (ii) expanding the terms in the Lagrange formula at $x = a$ and adding these values. In both cases the polynomial is written as a sum of **terms**, each of which is a product of easily found quantities. In the standard form, the terms are $c_i x^i$, while the terms of the Lagrange formula are numerical multiples of products of all but one of the $(x - x_i)$. Your report should show the **terms**, but need not show details of the multiplication leading to the terms. (Since you are computing the same function, the results should agree, but there should be differences that result from the different methods of computation.)

1c Solution Using the default accuracy of 10 significant figures in Maple, the values of the **terms** in the standard form of the polynomial, from highest degree to lowest, are

$$3,389,422.809;\ -13,586,974.42;\ 20,406,070.89;\ -13,608,873.27;\ 3,400,321.$$

The products on the lines of our expression in 1b for the Lagrange form of this **same** polynomial are

$$0.440\,627\,002\,8;\ -2.761\,939\,514;\ -32.646\,324\,30;\ 2.405\,306\,246;\ -0.430\,807\,464\,3$$

Maple shows only the digits that it believes to be accurately known (except in the case of the constant term, which is **exactly** 3400321) so this means that we can be confident of **only two** places after the decimal point in the standard expression while the Lagrange form will be accurate to **eight** decimal places.

Adding these terms, Maple gives -32.99 for the sum in standard form and -32.99313802 using the Lagrange form. As expected, since these are the same polynomial, these two agree to two decimal places, but more accuracy is only available using the Lagrange form. Other computing environments will **pretend** to full accuracy, but **roundoff errors** may have reduced the accuracy of computed values. Our error analysis has concentrated on **truncation error**, which concentrates on limitations of approximation methods assuming the ability to calculate with real numbers **exactly**. While it is usually possible to obtain the value of expressions as precisely as you like, it requires **planning** — just pushing a key on a calculator only gives values that are good for **immediate use**, but may not be very accurate when combined with other quantities.

It is important to retain the Lagrange interpolation polynomial in the form shown. All of the **factors** $(x - n)$ for n between 41 and 45 are known to 10 significant figures. When numbers are multiplied the result has **comparable relative accuracy** to the factors, so the products will retain 10 significant figures. Considering the size of the numbers, this gives 8 decimal places. Addition preserves the **absolute accuracy** of the terms in the sum, so we have the same 8 decimal places in the answer. In particular, if the terms in the Lagrange expansion are multiplied out, the resulting polynomials in standard form will have all of the weakness of the original expression. These standard forms must sum to the given expression, so, in each degree, at least one of the polynomials will have a large coefficient. However, we know that the values of the polynomials are fairly small, so there must be cancellation losing 6 digits of accuracy somewhere in the evaluation.

Problem 2

Although classical uses of interpolation start from a table of a function at equally spaced points, this is not required by the theorems. Even the divided differences used in Newton's interpolation formula make no essential use of the spacing of the points. Thus, if you want to build an interpolating polynomial for a function whose inverse is easier to find than the function itself, you can use the inverse function to build the table that leads to the divided difference table. We illustrate by building an interpolating polynomial for the inverse of $\sin x/x$ met in Problem 1 of Workshop 3. If we call this function q, then $v = q(u)$ for $0 \le u \le 1$ means $u = \sin v/v$ and $0 \le v \le \pi$. To get reasonable accuracy with only a few widely spaced points, we work in the middle of the domain of q.

2a Statement Start with $v = 1.00, 1.25, 1.50, 1.75, 2.00$ and find the corresponding values of u. Build a divided difference table by using the resulting values of u as the arguments and the given values of v as the values of $q(u)$ for these u.

2a Solution The usual way of writing a divided difference table with the differences arranged in columns according to their order gives a nice summary of the values, but is difficult to typeset and obscures the computation of the columns. The full notation for the divided differences requires listing all points in the domain contributing to the value of the divided difference. A useful compromise is to fix the order of the points (which need have no relation to their order on a number line) and designate the divided differences by the first (a) and last (b) domain point contributing to the value. The step from one divided difference table to the next builds the i^{th} row of the new table from the i^{th} row of the old table and the one below it. The new a_i is the old a_i and the new b_i is the old b_{i+1}. The column v containing the divided difference uses the expression $(v_{i+1} - v_i)/(b_{i+1} - a_i)$ to combine entries in the old table to get the new v_i. To maintain uniformity, the original table of function values will be written as a zeroth order divided difference table with $a_i = b_i$.

For the function in this problem, the function values are $v = 1.00, 1.25, 1.50, 1.75, 2.00$ and the domain needs to be computed because we are trying to construct a table for the **inverse** of $\sin(x)/x$. Thus our zeroth order table is

a	b	v
0.8414709848	0.8414709848	1.
0.7591876955	0.7591876955	1.25
0.6649966577	0.6649966577	1.5
0.562277683	0.5622776839	1.75
0.4546487134	0.4546487134	2.

The first order table is

a	b	v
0.8414709848	0.7591876955	−3.038283984
0.7591876955	0.6649966577	−2.654180332
0.6649966577	0.5622776839	−2.433824938
0.5622776839	0.4546487134	−2.322794679

Note that there is one fewer row in a new divided difference table. The second order table is

a	b	v
0.8414709848	0.6649966577	−2.176541247
0.7591876955	0.5622776839	−1.119066482
0.6649966577	0.4546487134	−.5278409512

The third order table is

a	b	v
0.8414709848	0.5622776839	−3.787607946
0.7591876955	0.4546487134	−1.941378824

The fourth order table is the single row

a	b	v
0.8414709848	0.4546487134	−4.772809785

2b Statement Write the fourth order Newton interpolation formula determined by these five points. Leave it in the form produced by the interpolation formula.

2b Solution The coefficients are the first entries in the v column and the new factor is $(x - b_1)$ from the old matrix. The zeroth order table contributes the constant term of the expression. Thus, the interpolating polynomial is the sum of

$$1.$$
$$-3.038283984(x - 0.8414709848)$$
$$-2.176541247(x - 0.8414709848)(x - 0.7591876955)$$
$$-3.787607946(x - 0.8414709848)(x - 0.7591876955)(x - 0.6649966577)$$
$$-4.772809785(x - 0.8414709848)(x - 0.7591876955)(x - 0.6649966577)(x - 0.5622776839)$$

2c Statement Use the expression obtained in (b) to find $q(.6)$ and $q(.7)$. Test these answers by evaluating $\sin x/x$ at the results.

2c Solution A useful feature of this expression is the ability to evaluate it using something like a **Horner scheme**. We trace these steps for the two given values of u

action	$v = 0.6$	$v = 0.7$
$c_4 = -4.772809785$	-4.77280978	-4.77280978
$(x - 0.5622776839)$	0.0377223161	0.1377223161
multiply last 2 lines	-0.1800414394	-0.6573224179
add $c_3 = -3.787607946$	-3.967649385	-4.444930364
$(x - 0.6649966577)$	-0.0649966577	0.0350033423
multiply last 2 lines	0.2578839490	-0.1555874190
add $c_2 = -2.176541247$	-1.918657298	-2.332128666
$(x - 0.7591876955)$	-0.1591876955	-0.0591876955
multiply last 2 lines	0.3054266337	0.1380333214
add $c_1 = -3.038283984$	-2.732857350	-2.900250663
$(x - 0.8414709848)$	-0.2414709848	-0.1414709848
multiply last 2 lines	0.6599057556	0.4103013175
add $c_0 = 1.$	1.6599057556	1.4103013175

The last entries in each column is the value of the function. To check this, recall that the function tabulated here is the **inverse function** of $g(x) = \sin(x)/x$. If we apply g to the result, we should get something close to the original values of 0.6 and 0.7. The computed values are 0.6000535739 and 0.6999555952, respectively. The error here is about 5×10^{-5}. The theoretical form of the error term is a **fifth order** divided difference (which is a **fourth derivative** divided by 4!) multiplied by the product of the **five** factors $(x - u_i)$. Because of the complicated nature of the function we are interpolating, it would be difficult to get the relevant derivative, but the values found so far in the divided difference computation suggest that it is likely to be of moderate size. The product of the other factors is about 10^{-5} in both cases. The use of $g(v)$ introduces another factor of the derivative of this function, which is about -0.4 for the values considered here so it has only a minor effect. The error seems consistent with what we expect.

End of Workshop 4