Mathematics 373 Workshop 9 Solutions

Pi and the AGM

Fall 2003

Introduction. The title of this workshop is borrowed from reference [1]. This book is the standard introduction to the techniques used for extremely high precision computation. The algorithms described here are also featured in the classical text [3]. A strikingly simple proof of the formula for the limit of this iteration, with a discussion of its deeper significance appears in [2]. Although it could have appeared earlier in the course, this workshop would have been a distraction while introducing topics needed for the midterm exam.

Problem 1. This problem deals with the calculation of π by Archimedes. Archimedes proved that

$$\frac{223}{71} < \pi < \frac{22}{7}$$

by calculating the **perimeters** of regular polygons of 96 sides **inscribed** and **circumscribed** about a unit circle. (Arc length for **convex** figures is **monotonic**, so that the inscribed polygon has perimeter less than 2π and the circumscribed polygon has perimeter greater than 2π). The reason for using 96 sides was that he started with a hexagon and applied a formula that related the perimeters of the polygons with n sides to those with $2n$ sides. You can do much better with your calculator (although you don't have to — the calculator has the value of π stored to the limit of its accuracy). In modern notation, the algorithm is simple.

1a Statement. **Trigonometry** gives that the perimeter I_n of an inscribed polygon of n sides and the perimeter O_n of a circumscribed polygon of n sides are

$$\begin{aligned} I_n &= 2n\sin\frac{\pi}{n} = 4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n} \\ O_n &= 2n\tan\frac{\pi}{n} = \frac{4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}{\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{2n}}. \end{aligned}$$

Introduce $\alpha_n = 1/O_n$ and $\beta_n = 1/I_n$ and show that $\alpha_n+\beta_n = 2\alpha_{2n}$ and $\alpha_{2n}\beta_n = \beta_{2n}^2$, leading to

$$\alpha_{2n} = \frac{1}{2}(\alpha_n+\beta_n) \quad \text{and} \quad \beta_{2n} = \sqrt{\alpha_{2n}\beta_n}$$

1a Solution. From the given results, we conclude that

$$\alpha_n = \frac{\cos\frac{\pi}{n}}{2n\sin\frac{\pi}{n}} = \frac{\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{2n}}{4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}$$

and

$$\beta_n = \frac{1}{2n\sin\frac{\pi}{n}} = \frac{1}{4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}} = \frac{\cos^2\frac{\pi}{2n}+\sin^2\frac{\pi}{2n}}{4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}.$$

Adding these gives

$$\alpha_n + \beta_n = \frac{2\cos^2\frac{\pi}{2n}}{4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}} = \frac{\cos\frac{\pi}{2n}}{2n\sin\frac{\pi}{2n}} = 2\alpha_{2n}.$$

Combining this with the intermediate form of β_n gives

$$\alpha_{2n}\beta_n = \frac{\cos\frac{\pi}{2n}}{4n\sin\frac{\pi}{2n}} \cdot \frac{1}{4n\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}} = \left(\frac{1}{4n\sin\frac{\pi}{2n}}\right)^2 = \beta_{2n}^2.$$

These are the desired results.

1b Statement. One has $\alpha_n < \beta_n$ for all n. In the doubling action of (a), the α's are increasing, the β's are decreasing, and they have a common limit that can be used to find the perimeter of the circle. If we start with a hexagon, $I_6 = 6$ and $O_6 = 4\sqrt{3}$, and the limiting values of I_n and O_n as $n \to \infty$ is 2π. In terms of our new variables, $\alpha_6 = \sqrt{3}/12 \approx 0.14434$ and $\beta_6 = 1/6 \approx 0.16667$.

Show that $\beta_n - \alpha_{2n} = (\beta_n - \alpha_n)/2$ and $\beta_{2n} - \alpha_{2n} \leq (\beta_n - \alpha_{2n})/2$, so that $\beta_{2n} - \alpha_{2n} \leq (\beta_n - \alpha_n)/4$.

1b Solution. Directly,

$$\beta_n - \alpha_{2n} = \beta_n - \frac{1}{2}(\alpha_n + \beta_n) = \frac{1}{2}(\beta_n - \alpha_n),$$

and

$$\beta_{2n} - \alpha_{2n} = \sqrt{\alpha_{2n}\beta_n} - \alpha_{2n} = \sqrt{\alpha_{2n}}\left(\sqrt{\beta_n} - \sqrt{\alpha_{2n}}\right) = \sqrt{\alpha_{2n}}\,\frac{\beta_n - \alpha_{2n}}{\sqrt{\beta_n} + \sqrt{\alpha_{2n}}}.$$

Now, $\alpha_n \leq \alpha_{2n} \leq \beta_n$. Hence, $\sqrt{\beta_n} \geq \sqrt{\alpha_{2n}}$, so

$$\frac{\sqrt{\alpha_{2n}}}{\sqrt{\beta_n} + \sqrt{\alpha_{2n}}} \leq \frac{1}{2},$$

as required. The final statement results from chaining these two inequalities.

1c Statement. Use this iteration to find the bounds on π from the 96 sided polygon and show that it implies the bounds attributed to Archimedes.

1c Solution. The values (to 9 decimal places) found by Maple are: $\alpha_6 = 0.1443375673$, $\beta_6 = 0.1666666667$, $\alpha_{12} = 0.1555021170$, $\beta_{12} = 0.1609876378$, $\alpha_{24} = 0.1582448774$, $\beta_{24} = 0.1596103662$, $\alpha_{48} = 0.1589276218$, $\beta_{48} = 0.1592686281$, $\alpha_{96} = 0.1590981250$, $\beta_{96} = 0.1591833537$. The upper bound on π is $1/(2\alpha_{96}) = 3.142714598$ and the lower estimate is $1/(2\beta_{96}) = 3.141031951$. To obtain good fractional approximations, the **continued fraction** is used. The calculation subtracts the integer part of a number and inverts to get a new quantity on which this process will be iterated. This gives

$$3 + \cfrac{1}{7 + \cfrac{1}{11.03852264}} \leq \pi \leq 3 + \cfrac{1}{7 + \cfrac{1}{143.0272636}}$$

The lower bound can be made smaller by replacing 11.03852264 by 10. Expanding this fraction gives $223/71$. In the upper bound, 143.0272636 can be replaced by ∞ to give the larger value $22/7$. It is not clear why Archimedes didn't use 11 instead of 10 in computing the lower bound. This would give 245/78, which is not significantly more complicated.

If this construction is performed with a more accurate value of π the result is

$$3 + \cfrac{1}{7 + \cfrac{1}{15.99659976}}$$

and using 16 as an upper bound on the last quantity gives 355/113 as an upper bound on π. It would take many more steps of the Archimedes method to obtain this upper bound. (I got as far as finding the equivalent of polygons of 6144 sides and had not yet obtained this upper bound. One more step does give this upper bound.)

Problem 2.

Instead of alternately calculating arithmetic and geometric means, Gauss investigated the calculation of the two means in parallel. That is, start with $\alpha > \beta > 0$ and set $\alpha' = (\alpha + \beta)/2$ and $\beta' = \sqrt{\alpha\beta}$. This makes a vast difference in the rate of convergence.

2a Statement.

Show that

$$\alpha' - \beta' < \frac{(\alpha - \beta)^2}{8\beta}$$

so that this operation is quadratically convergent.

2a Solution.

$$\alpha' - \beta' = \frac{\alpha + \beta}{2} - \sqrt{\alpha\beta} = \frac{\alpha + \beta - 2\sqrt{\alpha\beta}}{2} = \frac{(\sqrt{\alpha} - \sqrt{\beta})^2}{2}$$

Applying the mean value theorem for the function $f(x) = \sqrt{x}$, for which $f'(x) = 1/(2\sqrt{x})$ gives

$$\alpha' - \beta' = \frac{1}{2}\left(\frac{\alpha - \beta}{2\sqrt{\gamma}}\right)^2 = \frac{(\alpha - \beta)^2}{8\gamma}$$

for some γ between α and β. Since this expression is a **decreasing** function of γ, an **upper** bound is obtained by replacing γ by its **lower** bound β.

Note also that the expression for $\alpha' - \beta'$ is always positive, so we have $\alpha' > \beta'$.

2b Statement.

Continue this process to form a sequence of pairs $\left(\alpha^{(n)}, \beta^{(n)}\right)$. The common limit of the $\alpha^{(n)}$ and $\beta^{(n)}$ is called the **arithmetic-geometric mean** (or **AGM**, for short) of α and β. Gauss had computed 20 decimal places of the AGM of 1 and $\sqrt{2}$ in 1791. Find this value to the full accuracy of your calculator, and record the number of iterations required. How many did Gauss need to get 20 decimal places?

2b Solution.

My calculator is Maple. I set it to give 50 digits, but I will only show the results rounded to 15 places. The columns are β and α. The first row should be considered as row zero since no calculations have been performed on this given data.

1.	1.414213562373095
1.189207115002721	1.207106781186548
1.198123521493120	1.198156948094634
1.198140234677307	1.198140234793877
1.198140234735592	1.198140234735592

Four steps of the calculation got us to the point where $\alpha = \beta$ to the displayed accuracy.

Note, however, that after **three steps**, $\alpha - \beta$ is already less that 1.2×10^{-10}. Squaring this and dividing by 8 (this is an upper bound since $\beta > 1$) shows that the values after 4 steps will differ by at most 1.8×10^{-21}, so 20 place accuracy would be found by retaining that much accuracy in the calculation that we have already done. Thus Gauss needed only **four steps** to get a value of

$$1.19814\ 02347\ 35592\ 20744$$

My 50 place calculation shows that the last digit show above has been rounded up from 3 and gives the next 25 digits as

$$99224\ 92280\ 32387\ 82272\ 12663$$

2c Statement. In his diary entry for May 30, 1799, Gauss notes that the value of

$$\frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}},$$

that he computed at that time seemed to be the **reciprocal of** the number found in (b). Compute this integral by any method and compare to the result in (b).

2c Solution. The problem was incorrect as originally stated. The correction (that the two numbers are reciprocal rather than equal) has been incorporated into this version. (Also the dt has been restored to the integral.)

This integral is improper, so a direct numerical method will have difficulty with the fact that the value of the integrand at $t = 1$ is infinite. In order to get an integral that cam be evaluated numerically, we should write $t = \sin u$ with $dt = \cos u\, du$. The given integral is equal to

$$\frac{2}{\pi}\int_0^{\pi/2} \sqrt{\frac{\cos^2 u}{1-\sin^4 u}}\, du.$$

Since $\cos^2 u = 1 - \sin^2 u$ and $1 - \sin^4 u = (1 - \sin^2 u)(1 + \sin^2 u)$, the integral is

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{du}{\sqrt{1+\sin^2 u}}.$$

This integral can be evaluated by any direct method. For example, the composite trapezoidal rule with 2^n subintervals, $n = 0, 1, 2 \ldots$ gives

0.85355 33905 93273 76220 04221

0.83502 49857 60499 89746 64251

0.83462 70894 70804 99749 98348

0.83462 68416 74205 84319 18877

0.83462 68416 74028 96731 12771

0.83462 68416 74073 18628 14297

and the last agrees with the inverse of the AGM obtained above. In my 50 digit computation, the two values agreed to more than 45 places. This is shocking! The trapezoidal rule is only **second order**, yet the evaluation of the integrand at 17 points gives 13 digit accuracy and a mere 33 points suffices for more than 45 digits.

To explain this, recall the Euler-Maclaurin summation formula. This expresses the difference between an integral of a function f from a to b and a trapezoidal approximation as a sum of constant multiples of $f^{(k)}(b) - f^{(k)}(a)$ for all **odd** integers k. In this case, the integrand is $(1 + \sin^2 u)^{-1/2}$ and $\sin^2 u = (1 - \cos 2u)/2$, so the integrand is a function of $\cos 2u$. The first derivative of such a function is the product of a function of $\cos 2u$ with $\sin 2u$. Since the derivative of $\sin 2u$ is $2 \cos 2u$ and since $\sin^2 2u = 1 - \cos^2 2u$, the second derivative will be a function of $\cos 2u$. It follows that the odd derivatives are all $\sin 2u$ times a function of $\cos 2u$ and the even derivatives are all functions of $\cos 2u$. Our endpoints are 0 and $\pi/2$ and $\sin 2u = 0$ at both of these points. Thus **all main terms** in the Euler-Maclaurin summation formula are zero. The error for the trapezoidal approximation is found by stopping the asymptotic representation at a suitable place and considering the error term in the summation formula. The best stopping place will vary with the number of terms in the sum. Details would take us too far from this course, but our observation is no longer a mystery.

Comment. Gauss was eventually able to prove that these numbers were the same, and many other proofs have been given. The rapid convergence of the AGM shows that the integral in (c) can be computed quickly to high precision using the calculation in (b). The references explore this idea further.

References.

1 Jonathan M. Borwein & Peter B. Borwein, “Pi and the AGM”, Wiley, 1987.
2 B. W. Conolly, Solution of Problem 6672, *Amer. Math. Monthly* 100 (1993), 803.
3 John Todd, “Basic Numerical Mathematics, Vol. 1 Numerical Analysis”, Academic Press, 1979.

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