Mathematics 421 Essay 2
An operational view of Fourier coefficients
Spring 2008

0. Introduction

Although Laplace transforms were defined using an integral containing a parameter, the computation of transforms and inverse transforms used a special list of properties. This approach is generally known as an "operational" approach. Techniques of integration appear only where **necessary** to derive the properties that are taken as the basic properties of the transform. Although the properties of the transform (except for linearity) are as exotic as the rules of calculus, they become easy to use with a little practice.

Unfortunately, the treatment of Fourier coefficients in the text doesn't develop a similar list of properties, relying instead on classical techniques of integration. Each exercise becomes a separate calculation and results that provide structural hints to the form of the answer are hidden.

It is true that Fourier series don't always converge, in the usual **pointwise** sense, to the function that they represent, but the coefficients do tend to zero and the rate at which they decrease is related to the smoothness of the function. This provides a visual clue to what to expect from a computation of Fourier coefficients.

It is also true, to some extent, that term-by-term differentiation of the series corresponds to differentiation of the function. This provides an interpretation of some rules that is easier to remember than what was needed for the proof.

Part of the difficulty is that **real** Fourier series use trigonometric functions, so there are **two** functions for each positive index, and the term with index zero needs to be treated differently. The use of **complex exponentials** gives a single list of functions indexed by **all** integers, and the partial sums of the series are taken to be the sum of terms with index between $-n$ and $+n$ for integers n.

1. Real Fourier series

Given a function $f(x)$ on the interval $[-p, p]$, its **Fourier series** is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right)$$

where the **Fourier coefficients** a_n and b_n are defined by

$$\begin{aligned} a_0 &= \frac{1}{p} \int_{-p}^{p} f(x)\, dx \\ a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx \\ b_n &= \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx \end{aligned}$$

The **orthogonality** if the functions

$$1, \cos \frac{n\pi x}{p}, \sin \frac{n\pi x}{p}$$

assures us that the Fourier series of one of these functions is just the single term that is the function itself. In particular, the Fourier series of a constant function is itself.

Also note that the constant term of the series $a_0/2$ is the **average** value of $f(x)$.

Although it is customary to use only the **expression defining** $f(x)$ in describing exercises, the result depends **in an essential way** on the interval $[-p, p]$. The functions in the Fourier series for this interval are all **periodic** with period $2p$, so the actual function represented by the series is a **periodic extension** of fx. Different values of p usually lead to different extensions. Once the function is extended to be periodic, the integrals defining the Fourier coefficients can be computed using **any** interval of length $2p$. The use of a symmetric interval allows some saving in the case of **even functions** having only a **cosine series** or **odd functions** having only a **sine series**.

Another consequence of the need to use a periodic extension is that a function for which $f(p) \neq f(-p)$ must be considered as having a **jump discontinuity** at $x = p$.

2. Examples

Let

$$f(x) = \begin{cases} 1 & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

for some a with $0 < a < p$. This is an even function, so $b_n = 0$ for all n and

$$\begin{aligned} a_n &= \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx \\ &= \frac{1}{p} \int_{-a}^{a} \cos \frac{n\pi x}{p}\, dx \\ &= \frac{1}{p} \left[\frac{p}{n\pi} \sin \frac{n\pi x}{p} \right]_{-a}^{a} \\ &= \frac{2}{n\pi} \sin \frac{n\pi a}{p} \end{aligned}$$

for $n > 0$. We also have $a_0 = 2a/p$. Since this is a pure cosine series, the a_n computed here give the **Frequency Spectrum** defined in section 12.4 of the textbook. A special case of this example is Example 3 of that section, illustrated in Figure 12.19.

Let $g(x) = x$ on the interval $[-p, p]$. Except in the trivial case $p = 0$, we have $g(p) \neq g(-p)$, so this function must be considered as having a jump discontinuity at $x = p$. Note that the location of the discontinuity depends on p, so the common expression describes different function, with different series, for different p. This is an odd function, so all $a_n = 0$ and

$$\begin{aligned} b_n &= \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi x}{p}\, dx \\ &= \frac{1}{p} \int_{-p}^{p} x \sin \frac{n\pi x}{p}\, dx \\ &= \frac{1}{p} \left[\frac{p^2}{n^2\pi^2} \sin \frac{n\pi x}{p} - \frac{p}{n\pi} x \cos \frac{n\pi x}{p} \right]_{-p}^{p} \\ &= \frac{(-1)^{n+1} 2p}{n\pi} \end{aligned}$$

Details have been omitted, but it is easy to see that our expression for the indefinite integral is correct (we will describe a method of computing it later). To get the definite integral, we use $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for integer n.

Again, because this is a pure sine series, these coefficients give the Frequency Spectrum.

Note that both of these examples have Fourier coefficients that are a bounded quantity divided by n. Convergence of such series is often difficult to establish since **absolute convergence** can't be used.

3. Complex Fourier series

It is sometimes inconvenient to have two types of term, sines and cosines, appear in the series. This means that almost everything will require the discussion of several cases in proofs. The use of the identity

$$e^{ix} = \cos x + i \sin x$$

and its elementary consequences allows a unified approach to the theory at the expense of calculating with complex numbers. We will see that this means computing a single complex quantity $c_n = (a_n - b_n i)/2$ in place of the two real numbers a_n and b_n, the difference is mostly only a matter of bookkeeping. Although we will also require c_{-n} in order to describe the series, we have $c_{-n} = (a_n + b_n i)/2$, so the two complex numbers $c_{\pm n}$ are **complex conjugates** for every **real** function $f(x)$. We will need to justify the division by 2 and our choice of which conjugate to call c_n. This will appear in the course of describing how this is an equivalent formulation of the Fourier series.

Combining our definitions,

$$\begin{aligned} c_n &= \frac{1}{2p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx - \frac{i}{2p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)\big(\cos\frac{n\pi x}{p} - i\sin\frac{n\pi x}{p}\big)\,dx \\ &= \frac{1}{2p}\int_{-p}^{p} f(x)e^{-n\pi i x/p}\,dx \end{aligned}$$

Although $n > 0$ was assumed in this calculation, the result is valid for all n.

The computation of c_n using this expression applies to complex functions. If $f(x) = e^{m\pi i/p}$ for $m \neq n$, then

$$\begin{aligned} c_n &= \frac{1}{2p}\int_{-p}^{p} e^{m\pi i/p}e^{-n\pi i x/p}\,dx \\ &= \frac{1}{2p}\int_{-p}^{p} e^{(m-n)\pi i x/p}\,dx \\ &= \frac{1}{2p}\frac{p}{(m-n)\pi i}\big(e^{(m-n)\pi i} - e^{-(m-n)\pi i}\big) \\ &= 0 \end{aligned}$$

since the exponents differ by an integer multiple of 2π ($m \neq n$ was assumed when we wrote a formula requiring division by this quantity). When $m = n$, the integral reduces to the integral of the constant function 1 over an interval of length $2p$ divided by the length of the interval. so it is 1. Again, we see that each basic function, and hence each **linear combination** of basic functions, has a Fourier series that is itself.

4. A shifting theorem

Let functions of period $2p$ be related by $g(x) = f(x - b)$. We (temporarily) denote the complex Fourier coefficients of $f(x)$ by $[f]c_n$ and those of $g(x)$ by $[g]c_n$. Then

$$[g]c_n = \frac{1}{2p}\int_{x\in P} g(x)e^{-n\pi i x/p}\,dx$$

where P denotes any interval of length $2p$. Then,

$$\begin{aligned} [g]c_n &= \frac{1}{2p}\int_{x\in P} f(x-b)e^{-n\pi i x/p}\,dx \\ &= \frac{1}{2p}\int_{u+b\in P} f(u)e^{-n\pi i(u+b)/p}\,dx \\ &= \frac{e^{-n\pi i b/p}}{2p}\int_{u+b\in P} f(u)e^{-n\pi i u/p}\,dx \\ &= e^{-n\pi i b/p}[f]c_n \end{aligned}$$

A consequence of this is that f and g have the same Frequency Spectrum.

5. Differentiation

The expression x has derivative the constant function 1. However, in the context of Fourier series, this represents a **piece** of a periodic function having a single jump discontinuity. Its Fourier series primarily shows the signs of being influenced by this discontinuity. If a function has more than one jump discontinuity in a period, it can be written as a sum of a function taking only two values and a function with fewer jumps (see Exercise C).

If a periodic function is continuous except for at most a single jump continuity, we may choose an interval having the discontinuity at the endpoints to represent a period of the function and use this interval in computations.

Let us also assume that the function is **piecewise differentiable**; that is, the period interval can be written as a union of a finite number of closed intervals such that he function is differentiable on the interior of each of those intervals, and both **one-sided derivatives** exist at the endpoints. Visually, this says that the graph of the function is smooth except for a finite number of **corners**. The Fourier coefficients of this function will be a sum of integrals over each of the intervals on which the function is nice. On each of those intervals $[a,b]$, consider the **integration by parts** formula obtained from the derivative of $f(x)e^{-n\pi ix/p}$:

$$\int_a^b f'(x)e^{-n\pi ix/p}\,dx = \left[f(x)e^{-n\pi ix/p}\right]_a^b + \frac{n\pi i}{p}\int_a^b f(x)e^{-n\pi ix/p}\,dx.$$

The complex Fourier coefficients of $f'(x)$ are found by adding the terms on the left over all intervals $[a,b]$ and dividing by $2p$. In the first term on the right, each interval contribute the value of $f(x)e^{-n\pi ix/p}$ at the point with a positive sign if it is a right endpoint and a negative sign if it is a left endpoint. Our assumption of continuity says that all of these terms cancel except for the terms belong to the end of the period interval. The second terms on the right combine in the same way as the terms on the left to give the complex Fourier coefficients of $f(x)$ (ignoring the finite number of points at which it fails to exist) **multiplied by** $n\pi i/p$. This factor is exactly the factor expected from term-by-term differentiation of the Fourier series.

In the case of $g(x) = x$ on the interval $[-p, p]$, considered in Section 2, the derivative is the constant function 1, so its Fourier series consists only of this constant term, and this constant is completely determined by the change in the function over a period. Our previous values are easily obtained from this general formula: in fact, our computation followed this general computation.

6. Exercises

A. Use linearity and the results of examples in this document to find the Fourier series of $f(x) = 5x + 3$ on the interval $[-\pi, \pi]$.

B. Consider the following three functions on $[-\pi, \pi]$, each of which takes the value $+1$ $\frac{1}{6}$ of the time, -1 $\frac{1}{6}$ of the time, and 0 the remaining $\frac{2}{3}$ of the time:

$$f(x) = \begin{cases} 1 & 0 < x < \pi/3 \\ -1 & -\pi/3 < x < 0 \\ 0 & \text{otherwise} \end{cases}, \qquad g(x) = \begin{cases} 1 & \pi/3 < x < 2\pi/3 \\ -1 & -2\pi/3 < x < -\pi/3 \\ 0 & \text{otherwise} \end{cases},$$

$$h(x) = \begin{cases} 1 & \pi/3 < x < 2\pi/3 \\ -1 & -\pi/3 < x < 0 \\ 0 & \text{otherwise} \end{cases}.$$

Use linearity and the results of examples in this document to find the **complex Fourier series** and the **Frequency Spectrum** of each of these functions.

C. Consider the function on $[-1, 1]$ defined by

$$f(x) = \begin{cases} x & -1 < x \le 0 \\ 1 - x & 0 \le x \le 1 \end{cases} = \begin{cases} 0 & -1 < x \le 0 \\ 1 & 0 \le x \le 1 \end{cases} + \begin{cases} x & -1 < x \le 0 \\ -x & 0 \le x \le 1 \end{cases} .$$

The first expression shows two jump discontinuities in a period (at $x = 0$ and at $x = 1$). The second writes it as a sum of a function taking only two values and a continuous function. Since the second function is continuous, its Fourier series can be found by the method of Section 5. Use this to find the Fourier series of $f(x)$ on this interval.