Mathematics 421 Essay 1
Using the Laplace transform
Spring 2008

0. Introduction

The **Laplace transform** of a function of t is a function of a new variable s defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

This is an **improper** integral, so **convergence** must be considered. Typically, the integral will exist only for **sufficiently large** s, but **explicit** consideration of this restriction is usually not necessary. The functions that we will transform are covered by an **existence theorem** that guarantees that the integral exists for $s > a$ for a **piecewise continuous** function $f(t)$ with $|f(t)| < Ke^{at}$.

In section 4.1 of the textbook, an example was given in which this definition was easy to use:

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}.$$

The special case with $a = 0$ should be noted. However, it will turn out that even these examples are consequences of general properties of the transform, so that the definition will **only** be used to derive general properties. In stating these properties, only the simplest version will be shown; repeated application will be done as needed rather than used to state results with proofs requiring **mathematical induction**. Since the main applications involve **very few steps**, nothing is gained in most cases by pretending that there is a general formula.

1. Linearity

The most important property of the Laplace transform is **linearity**. This is a direct consequence of the linearity of integration. The basic statements are

$$\mathcal{L}\{f(t)\} = F(s) \implies \mathcal{L}\{c\,f(t)\} = c\,F(s)$$
$$\mathcal{L}\{f(t)\} = F(s) \text{ and } \mathcal{L}\{g(t)\} = G(s) \implies \mathcal{L}\{f(t)+g(t)\} = F(s)+G(s)$$

Repeated use of this rule deals with a sum of **arbitrarily many** terms, each of which is a product of a constant and a known function. The generalization to such expressions has been common since the first course in algebra. Such general expressions are called **linear combinations** of the known functions.

In addition to determining transforms, it will be necessary to find **inverse** transforms. Thus, any function that can be written as a linear combination of $1/(s-a)$ can be recognized as the Laplace transform of a linear combination of e^{at}. The method of **partial fractions** produces such an expression from **some** quotients of polynomials. Quotients of polynomials are called **rational functions**; and a rational function is called **proper** if the degree of the numerator is **strictly smaller** than the degree of the denominator. The functions that are Laplace transforms of linear combinations of exponentials are **proper** rational functions whose denominator is a product of distinct factors of the form $x - a$.

In the first course on Differential Equations, solutions of linear differential equations with constant coefficients were found by **assuming a solution** of the form $y = e^{at}$. Some equations had solutions that were trigonometric functions, and these could be found using Euler's identity $e^{it} = \cos t + i\sin t$. This leads to

$$\cos t = \frac{e^{it}+e^{-it}}{2} \quad\text{and}\quad \sin t = \frac{e^{it}-e^{-it}}{2i}$$

If we accept these formulas, then

$$\mathcal{L}\{\cos t\} = \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right) = \frac{s}{s^2+1}$$
$$\mathcal{L}\{\sin t\} = \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) = \frac{1}{s^2+1}$$

These formulas can also be obtained by characterizing the trigonometric functions as solutions of **initial value problems**. We illustrate this in section 3.

2. Derivatives

If you use **integration by parts** in the definition of $\mathcal{L}\{f'(t)\}$, you get

$$\int_0^\infty f'(t)e^{-st}\,dt = e^{-st}f(t)\big|_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = -f(0) + s\mathcal{L}\{f(t)\}$$

provided that $\lim_{t\to\infty} f(t)e^{-st} = 0$ for sufficiently large s (such $f(t)$ are said to be **of exponent order**, and this has already been assumed to assure the existence of the Laplace transform).

Theoretical aside.The existence of the Laplace transform of a function of exponential order is Theorem 4.2 of the textbook. Theorem 4.5 refines this method of proof to show that **the transform always approaches zero** as $s \to \infty$. In the case in which the transform is a rational function, this says that it is always a **proper** rational function. The partial fraction decomposition will then be a sum of **proper** partial fractions.

Repeated use of this formula gives expressions for the Laplace transform of derivatives of any order, but it is probably easier to invoke this formula twice to get an expression for $\mathcal{L}\{f''(t)\}$ than to remember the resulting formula, and derivatives of order higher than this will not usually be needed in this course.

An important application of this is the use of Laplace transforms to solve **initial value problems**. Suppose that $y(t)$ satisfies a **linear** differential equation with **constant coefficients** whose right side has a known Laplace transform, together with initial conditions at $t = 0$ that serve to define $y(t)$ uniquely. Assume that $\mathcal{L}\{y(t)\} = Y(s)$. Then, the Laplace transform of the left side of the equation is the product of a polynomial in s times $Y(s)$ plus another polynomial in s. This polynomial will be of lower degree than the coefficient of $Y(s)$. Equating this to the Laplace transform of the right side gives a linear **algebraic** equation for $Y(s)$. If the Laplace transform of the right side is a rational function, then $Y(s)$ will also be a rational function. It is also guaranteed to be **proper**. The solution of the initial value problem is reduced to **partial fractions** and some basic examples of Laplace transforms. This often leads to a better organization of **initial value problems**, but it gives the **same solution** that you would find by traditional methods. The factors of the denominator of $Y(s)$ are determined by the exponential functions that satisfy the corresponding homogeneous equation and the functions appearing on the right side. In case of repeated factors in the denominator of $Y(s)$, the Laplace transform method finds the polynomials multiplying exponential functions in the solution directly without the use of **undetermined coefficients**. The whole process is also considerably simpler than **variation of parameters**.

Linearity tells us that $\mathcal{L}\{0\} = 0$, so the rule for derivatives gives

$$0 = \mathcal{L}\{0\} = s\mathcal{L}\{1\} - 1$$

from which we conclude the previous result that $\mathcal{L}\{1\} = 1/s$. The transform of higher powers can be found by **mathematical induction** using the derivative formula for $f(t) = t^n$ which is

$$n\mathcal{L}\{t^{n-1}\} = s\mathcal{L}\{t^n\}$$

for $n > 0$.

Applying the rule that Laplace transforms tend to zero as $s \to \infty$ to $\mathcal{L}\{f'(t)\}$ tells us that

$$\lim_{s\to\infty} sF(s) = f(0)$$

This is an example of information about a function being **visible** in its Laplace transform.

3. Examples

The function e^{at} is the solution $y(t)$ of the **initial value problem**

$$\frac{dy}{dt} - ay = 0, \qquad y(0) = 1$$

If $\mathcal{L}\{y(t)\} = Y(s)$, then the initial condition gives $\mathcal{L}\{y'(t)\} = sY(s) - 1$, and the equation asserts that $(s-a)Y(s) - 1 = 0$. Thus, $\mathcal{L}\{e^{at}\} = 1/(s-a)$, as has already been noted.

Similarly, $\cos t$ is the solution $y(t)$ of the **initial value problem**

$$\frac{d^2y}{dt^2} + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

If $\mathcal{L}\{y(t)\} = Y(s)$, then the initial conditions give $\mathcal{L}\{y'(t)\} = sY(s) - 1$ and $\mathcal{L}\{y''(t)\} = s(sY(s) - 1) = s^2Y(s) - s$, and the equation asserts that $(s^2+1)Y(s) - s = 0$.

Another simple example is $\sin t$, which is the solution $y(t)$ of the **initial value problem**

$$\frac{d^2y}{dt^2} + y = 0, \qquad y(0) = 0, \qquad y'(0) = 1$$

If $\mathcal{L}\{y(t)\} = Y(s)$, then the initial conditions give $\mathcal{L}\{y'(t)\} = sY(s)$ and $\mathcal{L}\{y''(t)\} = s(sY(s)) - 1 = s^2Y(s) - 1$, and the equation asserts that $(s^2+1)Y(s) - 1 = 0$.

4. Scaling the argument

Consider functions $f(t)$ and $g(t)$ related by $g(t) = f(bt)$ for some constant b. Denote the transform of $f(t)$ by $F(s)$. Then, the substitution $u = bt$ gives

$$\begin{aligned}
G(s) = \mathcal{L}\{g(t)\} &= \int_0^\infty g(t)e^{-st}\,dt \\
&= \int_0^\infty f(bt)e^{-st}\,dt \\
&= \int_0^\infty f(u)e^{-su/b}\,\frac{1}{b}du \\
&= \frac{1}{b}\int_0^\infty f(u)e^{-(s/b)u}\,du \\
&= \frac{1}{b}F\big((s/b)\big)
\end{aligned}$$

As in this computation, the convention of using the corresponding upper case letter to name the Laplace transform of a function named by a lower case letter is used throughout this subject. Here are some examples of scaling:

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \frac{1}{a}\frac{1}{(s/a)-1} = \frac{1}{s-a} \\
\mathcal{L}\{\cos kt\} &= \frac{1}{k}\frac{1}{(s/k)^2+1} = \frac{k}{s^2+k^2} \\
\mathcal{L}\{\sin kt\} &= \frac{1}{k}\frac{s/k}{(s/k)^2+1} = \frac{s}{s^2+k^2}
\end{aligned}$$

This result also tells us the **form** of the transform of $f(t) = t^n$. If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\frac{1}{c^n}F(s) = \mathcal{L}\left\{\left(\frac{t}{c}\right)^n\right\} = cF(cs).$$

That is, multiplying s by c multiplies $F(s)$ by c^{-n-1}, so that $F(s)$ is of degree $-(n+1)$. Indeed, making the substitution $u = st$ in the integral defining $\mathcal{L}\{t^n\}$ gives the form of the transform and an expression for the constant factor for **arbitrary real** values of $n > 0$. Exercise 41 in section 4.1 asks for the details. This is also discussed in Appendix II . In the notation of these references to the textbook, we have

$$\mathcal{L}\{t^{\alpha-1}\} = \int_0^\infty t^{\alpha-1} e^{-st}\,dt = \int_0^\infty \left(\frac{u}{s}\right)^{\alpha-1} e^{-u}\,\frac{du}{s}$$

with $u = st$. Thus, the transform is $s^{-\alpha}$ multiplied by the constant

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u}\,du.$$

Integration by parts, as in the process for finding the Laplace transform of a derivative, shows that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ when both $\Gamma\alpha$ and $\Gamma(\alpha+1)$ exist. Assuming this property in general extends the definition of $\Gamma\alpha$ to all $alpha$ other than the negative integers.

5. Multiplying by an exponential

Suppose $g(t) = e^{at} f(t)$. Then

$$G(s) = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = F(s-a)$$

In particular,

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \quad \text{and} \quad \mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

Expand these formulas to simplify the expressions for the transforms. Use **completing the square** to return to this form when computing an **inverse Laplace transform**. In particular, combining this section with the previous one allows the Laplace transforms of all $t^n e^{at}\cos bt$ and $t^n e^{at}\sin bt$ to be found from the special cases in which $a = 0$ (and $b = 1$ if the trigonometric factor is present). Finding inverse transforms uses standard algebraic techniques to recognize the relation between the given expression and the transform of a simpler one.

The transforms of $t^n\cos t$ and $t^n\sin t$ for $n > 0$ require more care. Complex numbers provide the fastest route to formulas for the transforms.

$$\mathcal{L}\{t^n\cos t\} = \frac{1}{2}\left(\frac{1}{(s-i)^{n+1}} + \frac{1}{(s+i)^{n+1}}\right)$$
$$\mathcal{L}\{t^n\sin t\} = \frac{1}{2i}\left(\frac{1}{(s-i)^{n+1}} - \frac{1}{(s+i)^{n+1}}\right)$$

These fractions may be combined over a common denominator of $(s^2+1)^{n+1}$, but the numerators consist of the sum of alternate terms in the expansion of $(s \pm i)^{n+1}$. The **binomial theorem** gives an efficient computation of the terms, but there are still roughly $n/2$ terms in each of the numerators of the transforms.

This also leads to a difficulty when computing inverse transforms with these denominators because the simplest transforms are not in the form of **linear** expressions divided by powers of x^2+1 as produced by the usual partial fraction decomposition. In place of this, there is a particular polynomial of degree $2n$ that appears in the transform of $t^n\cos t$ or one of degree $2n+1$ that appear in the transform of $t^n\sin t$. When finding an inverse Laplace transform, one first determines the terms containing t^n to leave a problem of finding an inverse transform of a expression with a lower power of x^2+1 in the denominator. This could be done with an arbitrary quadratic factor, but it is probably better to use **completing the square** and **scaling** to reduce to the this special case before attempting this analysis.

When $n = 1$, the quadratic formula gives

$$\begin{aligned} \mathcal{L}\{t \cos t\} &= \frac{s^2 - 1}{(s^2 + 1)^2} = \frac{(s^2 + 1) - 2}{(s^2 + 1)^2} \\ &= \frac{1}{s^2 + 1} - \frac{2}{(s^2 + 1)^2} \\ \mathcal{L}\{t \sin t\} &= \frac{2s}{(s^2 + 1)^2} \end{aligned}$$

From this, we see that $\mathcal{L}\{\sin t - t \cos t\} = 2(s^2 + 1)^{-2}$.

6. Series

Assuming that the operations (essentially an interchange of limits) can be justified, the formula $\mathcal{L}\{t^n\} = n! s^{-n-1}$ leads to

$$\mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{a_n t^n}{n!}\right\} = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}$$

This allows **all** coefficients of the Taylor series for $f(t)$ about $t = 0$ to be related to those in a series expansion of $F(s)$ in powers of s^{-1} — which could be called a series expansion at infinity. The expansion of $f(t)$ is required to be **very strongly** convergent because of the $n!$ in the denominator, in order to allow the series for $F(s)$ to converge **anywhere**. However, these conditions are satisfied for the solutions of linear differential equations with constant coefficients that are the main examples used for $f(t)$. Laplace transforms are typically defined only for $s > c$ for some c, so a series for the transform should have the same property. This requires that the coefficients a_n should grow no faster than c^n.

In particular,

$$e^{at} = \sum_{n=0}^{\infty} a^n \frac{t^n}{n!}$$

so that

$$\begin{aligned} \mathcal{L} e^{at} &= \sum_{n=0}^{\infty} \frac{a^n}{s^{n+1}} \\ &= \frac{1}{s} \sum_{n=0}^{\infty} \left(\frac{a}{s}\right)^n \\ &= \frac{1}{s} \frac{1}{1 - \frac{a}{s}} = \frac{1}{s - a} \end{aligned}$$

An interesting example is the **Bessel function** of order zero. This function satisfies

$$t y'' + y' + t y = 0.$$

This equation is singular, but the **method of Frobenius** assures us that there is a unique solution with $y(0) = 1$, and it gives a series for this solution that converges for all t. In fact, there is a recurrence for the coefficients that allows the coefficient of t^n to be compared to $1/n!$.

The coefficients of this differential equation are polynomials in t and not constant, so the formula to be discussed in Section 19 below produces a differential equation for the Laplace transform of this function. That equation turns out to be **first order**, so there is a standard method to get a closed form solution.

On the other hand, the recurrence determining the terms of the series for $y(t)$ can also be used to find the terms of a series for $Y(s)$. It is an interesting exercise to show that the closed form solution for $Y(s)$ is represented by the series found in this way.

7. Partial fractions

If a Laplace transform is a rational function has a denominator $\prod(s-\alpha_i)$, with n factors, then the numerator has degree less than n because the function must have limit zero as $s \to \infty$. Such rational functions are called **proper** by analogy to arithmetic proper fractions that have numerators that are smaller than their denominators. The inverse Laplace transforms of such expressions can be found using the **partial fraction decomposition** that was used to integrate such expressions. In the case in which all factors of the denominator are **different linear factors** the decomposition is easily found by the method that the text calls the "cover up method" (page 205). In this method, a proper rational function is written as a sum of simpler expressions **of the same type**:

$$\frac{P(s)}{(s-a)Q(s)} = \frac{A}{(s-a)} + \frac{P_1(s)}{Q(s)} \qquad (*)$$

where $Q(a) \neq 0$ (indicating that $(s-a)$ does not divide $Q(s)$), and all fractions are proper fractions. Multiplying by $(s-a)$ and evaluating at $s=a$ gives

$$A = \frac{P(a)}{Q(a)}. \qquad (**)$$

This works because a proper fraction with a linear denominator has a constant numerator.

The partial fraction decomposition is a special case of the fact that, if $Q_0(x)$ and $Q_1(x)$ are polynomials over a field (an algebraic system like the real numbers, complex numbers or rational numbers that allows addition, multiplication and division — except that division by zero is not allowed) have no common factors (other than constants), then there are polynomials A_0 and A_1 (with coefficients in the same field) such that

$$A_1 Q_0 + A_0 Q_1 = 1.$$

The proof uses the **Euclidean algorithm**, which also gives an efficient computation A_0 and A_1. If the degree of A_0 in such an equation is of the same degree as Q_0 or larger, then one can subtract a certain multiple of Q_0 from A_0 and add the same multiple of Q_1 to A_1 to get other choices of A_0 and A_1 with the same property and having the degree of A_0 less than the degree of Q_0. In this case, the degree of A_1 will automatically be less than the degree of Q_1. If this equation is divided by Q_0Q_1, the result is a partial fraction decomposition of $1/(Q_0Q_1)$ as a sum of proper fractions whose denominators are Q_0 and Q_1.

This equation also gives

$$(PA_1)Q_0 + (PA_0)Q_1 = P.$$

for any polynomial P. If the degree of P is less than the degree of Q_0Q_1, the use of division to replace PA_0 by a polynomial of degree less than the degree of Q_0, with a corresponding change in the other term, leads to a partial fraction decomposition of any proper fraction with denominator Q_0Q_1 as a sum of a proper fraction of denominator Q_0 and a proper fraction with denominator Q_1. All that is needed is that Q_0 and Q_1 have no common factor. In the simplest case of the Euclidean Algorithm, one gets

$$1 \cdot (x-a) + (-1) \cdot (x-b) = b-a.$$

The general method and the cover-up method are almost identical in this case.

Note that each linear factor of the denominator is determined separately by this method. If there are any irreducible quadratic factors or repeated linear factors in the denominator, they can be left until all simple linear factors have been removed by using $(**)$ to identify the numerator in the first term on the right side of $(*)$ for each simple linear factor and subtracting that term to leave a simpler fraction. In finding the numerators for each factor of the denominator, the original numerator $P(x)$ may be used for all factors with an appropriate choice of $Q(x)$ for each factor.

Linear fractions of higher multiplicity can be handled by the following variant on (∗)

$$\frac{P(s)}{(s-a)^k Q(s)} = \frac{A}{(s-a)^k} + \frac{P_1(s)}{(s-a)^{k-1} Q(s)}$$

Here, multiplication by $(s-a)^k$ and evaluating at $s = a$ again gives (∗∗) although a factor of $(s-a)^{k-1}$ remains in the denominator. This allows a linear factor of multiplicity k to be removed in k steps, provided that the complementary term is found as part of each step.

If there is only one quadratic factor, the terms belonging to the other factors of the denominator can be found are subtracted from the original expression. When common factors are removed, the result will have only this factor in the denominator. However, there seems to be no easy way to deal with more than one quadratic factor.

8. Systems

A system of differential equations with the same number of equations as there are functions to be determined, together with suitable initial conditions, can be solved by applying Laplace transforms to replace the system by a system of linear algebraic equations for the transforms of the solutions. In many cases, a solution of these equations can be found directly. This requires no modification of the system prior to applying the Laplace transform. For example, if there are only two equations (and only two unknown functions), the simple formula for the inverse of a 2-by-2 matrix gives a formula for the Laplace transform of the solution with little effort. The study of the **double pendulum** gives a striking example of this method.

9. An example

Exercise 10 in section 4.6 asks to use Laplace transforms to solve

$$\begin{aligned} \frac{dx}{dt} - 4x + \frac{d^3y}{dt^3} &= 6\sin t \\ \frac{dx}{dt} + 2x - 2\frac{d^3y}{dt^3} &= 0 \end{aligned}$$

with initial conditions $x(0) = y(0) = y'(0) = y''(0) = 0$. If $\mathcal{L}\{x(t)\} = X(s)$ and $\mathcal{L}\{y(t)\} = Y(s)$, then the initial conditions give $\mathcal{L}\{x'(t)\} = sX(s)$ and $\mathcal{L}\{y'''(t)\} = s^3Y(s)$, so Laplace transform of the equations may be written in the simple matrix form

$$\begin{bmatrix} s-4 & s^3 \\ s+2 & -2s^3 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \frac{1}{s^2+1} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \frac{6}{s^2+1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The determinant of the coefficient matrix is

$$s^3\big(-2(s-4) - (s+2)\big) = -3s^3(s-2)$$

and the cofactor expression for the inverse gives

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \frac{2}{s^3(s-2)(s^2+1)} \begin{bmatrix} 2s^3 \\ s+2 \end{bmatrix}$$

Thus,

$$X = \frac{4}{(s-2)(s^2+1)} = \frac{4/5}{s-2} - \frac{4/5\,(s+2)}{s^2+1}$$

where the first term is found by the cover-up method, and the second by simplifying the difference of the left side and the first term, which is

$$\frac{20 - 4(s^2+1)}{5(s-2)(s^2+1)} = \frac{16 - 4s^2}{5(s-2)(s^2+1)} = \frac{-8+4s}{5(s^2+1)}$$

From the basic transform pairs, we get

$$\begin{aligned} x(t) &= \frac{4}{5}e^{2t} - \frac{4}{5}\cos t - \frac{8}{5}\sin t \\ x'(t) &= \frac{8}{5}e^{2t} - \frac{8}{5}\cos t + \frac{4}{5}\sin t \\ y'''(t) &= \frac{8}{5}e^{2t} - \frac{8}{5}\cos t - \frac{6}{5}\sin t \end{aligned}$$

where the first line is the inverse transform of $X(s)$, the second line is found by differentiating $x(t)$ and the third line is the common value found by solving each equation in the original system algebraically for $y'''(t)$. This checks that $x(t)$ is a solution of the differential equations, and it is easy to see that it satisfies $x(0) = 0$. From this, one could integrate — keeping track of the initial conditions — to find $y(t)$. However, we want to illustrate the use of partial fractions to write

$$Y = \frac{2s+4}{s^3(s-2)(s^2+1)} = \frac{A(s)}{s^3} + \frac{b}{s-2} + \frac{C(s)}{s^2+1}$$

with A of degree 3 and C of degree 1. The cover up method produces terms of

$$\begin{aligned} \frac{1}{5(s-2)} &= \frac{s^3(s^2+1)}{5s^3(s-2)(s^2+1)} \\ \frac{-2}{s^3} &= \frac{-10(s-2)(s^2+1)}{5s^3(s-2)(s^2+1)} \end{aligned}$$

With the denominator of the expressions on the right, including the numerical constant 5, the original numerator of $Y(s)$, found in the matrix solution, is $10s+20$. Subtracting the two numerators above from this gives $-s^5 + 9s^3 - 20s^2 + 20s$. This is clearly divisible by s, and it must be divisible by $s-2$ if our work is correct. Division reveals it to be $s(s-2)(-s^3 + 2s^2 + 5s - 10)$. The unidentified partial fractions add to

$$\frac{-s^3 - 2s^2 + 5s - 10}{5s^2(s^2+1)}$$

Another application of the cover up method gives a term of $-2/s^2 = -10(s^2+1) \,/\, \big(5s^2(s^2+1)\big)$. Subtracting this leaves $(-s^2 + 8s + 5) \,/\, \big(5s(s^2+1)\big)$. After one more step, we have the full partial fraction decomposition

$$Y(s) = \frac{1}{5}\frac{1}{s-2} - \frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s} + \frac{8-6s}{5(s^2+1)}$$

Taking inverse transforms gives

$$y(t) = \frac{1}{5}e^{2t} + 1 - 2t - t^2 + \frac{8}{5}\sin t - \frac{6}{5}\cos t.$$

From this, it is easy to obtain values of y', y'', and y''' to check all initial conditions and the previously discovered value of y'''.

10. The derivative of a transform

Assuming the validity of differentiating with respect to the parameter s under the integral sign, one has

$$F'(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty -tf(t)e^{-st}\,dt = -\mathcal{L}\{tf(t)\}.$$

In particular,

$$\mathcal{L}\{t\sin t\} = -\frac{d}{ds}\frac{1}{s^2+1} = \frac{2s}{(s^2+1)^2}$$
$$\mathcal{L}\{t\cos t\} = -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2}$$

This agrees with the results obtained using partial fractions over the complex numbers. Neither approach provides a simple way to evaluate inverse transforms of expressions with denominator $(s^2+1)^n$ for large values of n. Symbolic calculation systems like Maple have routines for handling this case, but there may not be a suitable general method for hand computation. The only expressions that are suitable for hand computation are of low enough degree that there are few opportunities for complicated expressions to arise.

11. Functions defined by cases

There is one more formula for Laplace transforms that is part of the general toolbox. If a function of t is zero for $t < a$ and given by some formula for larger t, the Laplace transform integral is best evaluated by the substitution $t = a + u$. This introduces a factor of e^{-as} and changes the integral to an integral in u from zero to infinity. This integral is the ordinary Laplace transform of the expression for our function in terms of u. To write this as a formula, we introduce the **Heaviside function** $\mathcal{U}(t-a)$, defined by

$$\mathcal{U}(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \ge a \end{cases}$$

Then, for $a \ge 0$,

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s).$$

The proof of this formula consists of rewriting the definition of $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}$ as follows

$$\begin{aligned}\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} &= \int_0^\infty f(t-a)\mathcal{U}(t-a)e^{-st}\,dt \\ &= \int_a^\infty f(t-a)e^{-st}\,dt \\ &= \int_0^\infty f(u)e^{-s(a+u)}\,du = e^{-as}F(s)\end{aligned}$$

The first step uses the definition of $\mathcal{U}(t-a)$ to restrict the domain of integration; then a new variable is introduced that goes from 0 to ∞ on this domain; and the result is interpreted in terms of the known transform of f. Note that there is an implicit factor of $\mathcal{U}(t)$ whenever we are taking a Laplace transform since only values of $t > 0$ are considered when evaluating the integral in the definition of the Laplace transform.

Any function defined by cases may be written in an equivalent form using Heaviside functions. If the expression defining the function changes at $t = a$, a term is introduced with factor of $\mathcal{U}(t-a)$ multiplying the **change** from the expression used for $x < a$ to the one used for $x > a$.

To retrieve a definition by cases from one using Heaviside functions, the value in each interval is the sum of the expressions multiplying $\mathcal{U}(t-a)$ with smaller values of a.

When using this formula to find Laplace transforms, one must be careful to express the quantity multiplying $\mathcal{U}(t-a)$ as a function of $t-a$. The alternate formula in formula (6) on page 213 of the text **should not be used**. It is always dangerous to try to work with two formulas that are minor variations of one another since you can wind up with a meaningless mix of parts of each formula. In most cases, it is easy to see how to write the expression multiplying $\mathcal{U}(t-a)$ in terms of $t-a$, so there is no need to try to develop a formula for the process.

When finding inverse transforms, terms with the same e^{-as} factor are collected together and the inverse transforms of each of these clusters is found separately. In applications to differential equations, this usually finds a **continuous solution** even when there is a **discontinuous driving force**.

A convenient way to work with this is to use a graphical description of the function. If the graph of $f(t)$ if known, then the graph of $f(t-a)\mathcal{U}(t-a)$ is found by translating the known graph a units to the right.

As an example, consider

$$f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ 2-t & \text{if } 1 < t < 2 \\ 0 & \text{if } t > 2 \end{cases}$$

whose graph consists of line segments from $(0,0)$ to $(1,1)$, from $(1,1)$ to $(2,0)$ and then a **ray** along the horizontal axis to the right.

Instead of working with formulas, we use a graphical method of finding $F(s)$ starting from the function $g(t) = t$ with $G(s) = 1/s^2$. Translating this graph one unit to the right gives a parallel line representing the graph of $g(t-1\mathcal{U}(t-1)$ having a Laplace transform of e^{-s}/s^2. Subtracting the second function from the first gives a line from $(0,0)$ to $(1,1)$ followed by a horizontal ray. The Laplace transform of this function is $\left(1-e^{-s}\right) / s^2$. Translating **this graph** one unit to the right gives a line from $(1,0)$ to $(2,1)$ followed by a horizontal ray. The Laplace transform of this function is $e^{-s}\left(1-e^{-s}\right) / s^2$. Subtracting the second function from the first gives the desired function and shows that its Laplace transform is $\left(1-e^{-s}\right)^2 / s^2$.

An extension of this method invents **generalized functions** like the **Dirac delta function** $\delta(t-a)$ that acts like a **derivative of the Heaviside function**. Physically, it plays the role of an **impulse** that effects an abrupt change in **momentum** when the terms in the equation represent **forces**. Its Laplace transform is e^{-as}. It may be used formally in a solution of differential equations by Laplace transforms and gives rise to a continuous solution of the equation. When made rigorous, this shows that any physically realizable force approximating an impulse leads to motion approximating this solution.

12. Convolution

There is one more operation inspired by the study of Laplace transforms. The **convolution** of $f(t)$ and $g(t)$, denoted by $(f*g)(t)$ is defined by

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

A change of variables in the sector defined by $0 < \tau < t < \infty$ in the (τ, t) plane shows that $\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$. This is the subject of theorem 4.9 of the text.

The case where $g(t) = 1$ gives

$$(f*g)(t) = \int_0^t f(\tau)\,d\tau.$$

Since $G(s) = 1/s$ in this case, we have

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}.$$

Note that the function on the left side of this equation has derivative $f(t)$ by the **fundamental theorem of calculus** and is zero at $t = 0$. Hence, this special case is an alternate form of the formula for the transform of a derivative.

An application to the **Volterra integral equation** in Section 4.4 (page 222) is a striking use of the idea of convolution.

Little more needs to be added to the treatment in the textbook except to note that it may be easier the find the inverse transform of a product of two transforms using partial fractions than to compute a convolution directly.

12. Summary

Note that more than one rule may apply to a given expression. Since all rules are consequences of the definition, you can be sure that any **correct** application of the rules will determine the same expression for the Laplace transform or its inverse. The availability of alternate methods of computation should be used as an opportunity to check your work and your understanding of all the rules for working with Laplace transforms.