Mathematics 421 Essay 5a
Vector Calculus (part 1)
Spring 2009

0. Introduction

In our multivariable calculus course, there is a chapter on vector calculus covering the **gradient** of a function of several variables, the **curl** and **divergence** of vector fields in three dimensions, and some integrals that invert these differentiation operators. The different versions of the **fundamental theorem of calculus** are treated separately, which emphasizes the technical differences instead of the relation to the fundamental theorem. Chapter 9 of the present text also gives many details. Another source of information about Vector Calculus is "div, grad, curl, and all that" by h. m. schey (the title page really is written entirely in lower case), fourth edition published by W. W. Norton, New York, 2005 (ISBN 0-393-92516-1). Another complete exposition is not needed; these notes only aim to introduce a collection of exercises illustrating what we need to express the **Laplacian** in order to work with the classical partial differential equations in coordinate systems other that the usual **rectangular** or **Cartesian** coordinates. The appearance of the Laplacian in these equations is based on geometric considerations, with the role of coordinates being only to allow it to be evaluated. Thus, it **must** change its appearance in different coordinate systems.

1. The del operator

In two or three dimensions, formal properties of the differentiation operators are expressed using

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

which operates on the function written to its right. When applied to a scalar function f, it gives a vector ∇f, called the **gradient** of f. If $u = f(x, y, z)$, then

$$\nabla f = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle$$

Such vector functions are often call **vector fields**. In addition, the rules used to define vector products give vector operators that have suggestive notation. For a vector field

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle,$$

the **curl** of $\mathbf{F}$ is the vector field

$$\nabla \times \mathbf{F} = \left\langle R_y - Q_z, P_z - R_x, Q_x - P_y \right\rangle,$$

with subscripts used to denote partial derivatives. The **divergence** of $\mathbf{F}$ is the scalar function

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z,$$

again with subscripts used to denote partial derivatives.

A great observation is that the fundamental rule

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

tells us that $\nabla \times \left(\nabla f \right) = 0$ and $\nabla \cdot \left(\nabla \times \mathbf{F} \right) = 0$.

In two dimensions, a formula defining $f(x, y)$ can be thought of as defining a function of x, y and z, that just happens to be independent of z. The gradient of f has third coordinate zero and first two coordinates giving a vector field in the plane. The curl of a vector field that is independent of z with third coordinate zero has first two coordinates zero, so that its third coordinate can be considered as a scalar derivative of a two dimensional vector field. It can be related to the divergence by noting that, when applied to $\mathbf{F}$, it is the divergence of a vector field obtained by rotating $\mathbf{F}$ through an angle of $\pi/2$ (clockwise).

In higher dimensions, the role of the cross product must be replaced by operators involving things that might be called multivectors. This is mentioned only as a reminder that special techniques used in low dimensions may need to be modified for problems in higher dimensions.

2. Line integrals

If a **force** is represented by a **vector field** (in any number of dimensions), the **work** done by that force in moving an object along a path $\mathcal{P}$ is computed by an integral

$$\int_{\mathcal{P}} \mathbf{F} \cdot \mathbf{ds} = \int_{\mathcal{P}} \mathbf{F} \cdot \mathbf{T}\, ds$$

computed by setting $\mathbf{ds} = \mathbf{T}\, ds = \langle dx, dy, \ldots \rangle$ on the curve $\mathcal{P}$. If $\mathcal{P}$ is given by expressing $x, y, \ldots$ as functions of a **parameter** t, then $dx = (dx/dt)dt$ as usual to give an integral in terms of t. The **chain rule** shows that **any** parameterization may be used to compute the integral, since all will give the same answer.

If $\mathbf{F} = \nabla f$, the **multivariable** chain rule shows that the integral is the difference of the values of f at the endpoints of $\mathcal{P}$, so it is "independent of path". In particular, the integral is zero if $\mathcal{P}$ is a **closed curve**.

Some convenient parameterizations that can be used to evaluate line integrals are:

(i) **line segments**. The line segment from (x_0, y_0) to (x_1, y_1) is given by $x = x_1 t + x_0(1-t)$, $y = y_1 t + y_0(1-t)$for $0 \le t \le 1$;

(ii) the **ellipse** $(x/a)^2 + (y/b)^2 = 1$ is given by $x = a\cos t$, $y = b\sin t$ for $0 \le t \le 2\pi$.

A quantitative form of the independence of path for ∇f is given by **Green's theorem**:

$$\oint_{\mathcal{C}} P\, dx + Q\, dy = \iint_{\mathcal{D}} Q_x - P_y\, dA$$

where $\mathcal{C}$ is a curve tracing the boundary of the region $\mathcal{D}$ in the counterclockwise sense, and dA signifies integration with respect to **area**. In rectangular coordinates, $dA = dx\, dy$. For simple regions the proof consists of noting that integrating Q_x with respect to x gives the difference of the values of Q at the endpoints and integrating this result with respect to y is exactly how one computes the path integral of $Q\, dy$, and similarly for the other terms on each side of this formula. Less simple regions need to be cut into simpler pieces. If a region is cut by a line, the integral over the whole region is the sum of the integrals over the pieces. To get closed curves, the pieces of the path integral around the boundary need to be completed with integrals along the cut, but one adds an integral in one direction to one piece and its reverse to the other piece. The sum of these new terms is zero.

Green's theorem can be used to **define** the integral over the region $\mathcal{D}$ with respect to area in terms of the conceptually simpler path integral around the boundary. One application is a change-of-variables formula for double integrals. It can also be used in the other direction to evaluate path integrals as double integrals with an integrand in which a great deal of redundancy has been removed. In particular, we know that the integral of 1 with respect to area is just the area of the region, and the integral of x is the moment with respect to the y axis, which is the product of the area with the x coordinate of the centroid. For simple regions, these are all known on the basis of computations done many times in earlier courses.

3. Changes of variables

Suppose that x and y are functions of another pair of variable u and v. If z is a function of x and y, a multivariable chain rule gives the derivatives of z with respect to the new variables by a formula that can be written in matrix form as

$$\begin{bmatrix} z_u \\ z_v \end{bmatrix} = \begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix} \begin{bmatrix} z_x \\ z_y \end{bmatrix}$$

This expression can be **inverted** to give

$$\begin{bmatrix} z_x \\ z_y \end{bmatrix} = \frac{1}{x_u y_v - x_v y_u} \begin{bmatrix} y_v & -y_u \\ -x_v & x_u \end{bmatrix} \begin{bmatrix} z_u \\ z_v \end{bmatrix}$$

which tells how to directly construct the expressions obtained by evaluating partial derivatives with respect to the old variables, or combining them into a gradient, and substituting expressions in terms of the new variables.

A benefit of the use of matrix notation is that the same formula may be interpreted in different ways: the product of a matrix M and a vector v may be considered as a vector of linear combinations of elements of v whose coefficients are the entries of M or a linear combination of the columns of M whose coefficients are the entries of v. The latter interpretation leads to a chain rule:

$$\nabla f = \nabla u \frac{\partial z}{\partial u} + \nabla v \frac{\partial z}{\partial v}$$

where $z = f(x, y)$ that is expressed in terms of the new variables u and v by the change of variables formula. This is particularly useful where the function f itself has not given, but one has the result of expressing its result in terms of the new variables. The expressions for ∇u and ∇v in terms of u and v are the columns of the inverse of the change of variables matrix that we constructed.

An important example is given by **polar coordinates**, in which the new variables are usually denoted r and θ with $x = r \cos\theta$ and $y = r \sin\theta$. Then, $x_r = \cos\theta$, $y_r = \sin\theta$, $x_\theta = -r \sin\theta$, and $y_\theta = r \cos\theta$. Then $x_r y_\theta - x_\theta y_r = r$ and

$$\nabla r = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \text{ and } \nabla\theta = \frac{1}{r} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

A second part of these notes will appear soon

4. Exercises

A. Given $f(x, y) = \ln(x^2 + 3y^2)$, find ∇f. Note: $f(x, y)$ is undefined when $(x, y) = (0, 0)$.

B. Given A: $(0, 0)$, B :: (3.0), C: $(3, 1)$, the boundary of the triangle ABC is traced in the counterclockwise direction by the three intervals AB, BC, CA.

(a) use this description to find

$$\oint_{\mathcal{C}} (2xy - 4y)\, dx + (2x + 1)\, dy$$

around the boundary $\mathcal{C}$ of triangle ABC.

(b) Let $\mathcal{T}$ denote triangle ABC, and apply Green's theorem to write this integral as a double integral with respect to area over $\mathcal{T}$.

(c) Since we have formulas for the area of a triangle and the location of its centroid, we know

$$\iint \mathcal{T}\, dA = \frac{3}{2}, \quad \iint \mathcal{T}\, x\, dA = 3, \quad \iint \mathcal{T}\, y\, dA = \frac{1}{2}.$$

Use this to evaluate the integral in (b). If this does not agree with the value in (a), there is a mistake somewhere. Correct any such mistake before considering the exercise complete.

C. Green's theorem shows that the area of a region with boundary $\mathcal{B}$ is given by

$$\frac{1}{2} \oint_{\mathcal{B}} x\, dy - y\, dx$$

Suppose that $\mathcal{B}$ is given in polar coordinates by an equation $r = g(\theta)$ for $0 \le \theta \le 2\pi$. Determine the result of using this to express the area as an integral with respect to θ. The result should look familiar.

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