Mathematics 421 Essay 3
Boundary Value Problems
Spring 2010

0. Introduction

A **second order** differential equation has a general solution containing two parameters. Typically these parameters are the values of the solution and its first derivative at a single point. Under suitable conditions, the theory predicts that such data leads to a unique solution. However, some natural questions lead to the value of the function at **two different points** being specified. In such questions, the function restricted to **the interval between those points** is the main object of interest, so these questions are called **boundary value problems**. If the equation is linear and homogeneous, and the given boundary values are zero, then a unique solution could only be the zero function. However, **there is no uniqueness theorem for boundary value problems**. Indeed, certain homogeneous equations with zero boundary values have non-trivial solutions.

1. The main example

A typical example is

$$\frac{d^2 y}{dx^2} + \lambda y = 0; \, y(0) = 0; \, y(L) = 0$$

for fixed L and a parameter λ. **Case I**: If $\lambda < 0$, write $\lambda = -\alpha^2$. Then, the general solution of the differential equation is $y = ae^{\alpha x} + be^{-\alpha x}$. The condition at $x = 0$ requires $b = -a$, so the solution is a multiple of $\sinh x$. This function is strictly increasing, so the condition at $x = L$ allows only the zero function. **Case II**: If $\lambda = 0$, the solution is $y = a + bx$, the condition at $x = 0$ gives $a = 0$, and again the solution is a multiple of the increasing function $y = x$, and only $b = 0$ allows $y(L) = 0$. **Case III**: If $\lambda > 0$, write $\lambda = \alpha^2$. Then, the general solution of the differential equation is $y = a \cos \alpha x + b \sin \alpha x$, and the condition at $x = 0$ gives $a = 0$. However, this time, if $\alpha = n\pi/L$, giving $\lambda = n^2 \pi^2 / L^2$, all multiples of $\sin \alpha x$ satisfy the condition at $x = L$.

It is only a coincidence that the sign of λ appears significant in this analysis. What is important is the form of the solution of a second order linear differential equation with constant coefficients. After testing for solutions of the form $e^{\alpha x}$, one distinguishes the cases of **real α**, **complex α**, and **repeated roots**. These cases reflect the nature of the solution. The boundary condition at $x = 0$ selects the multiples of a single solution for each λ, and this solution is tested for satisfying the boundary condition at $x = L$.

The functions that appear in these solutions are exactly the **odd functions** on $-L \le x \le L$ appearing in **half range Fourier series**. That is, if the given $f(x)$ on $[0, L]$ is **extended** to be **odd** and periodic with period $2L$, the Fourier series of the extended function is identical with the eigenfunction expansion of the original function $f(x)$.
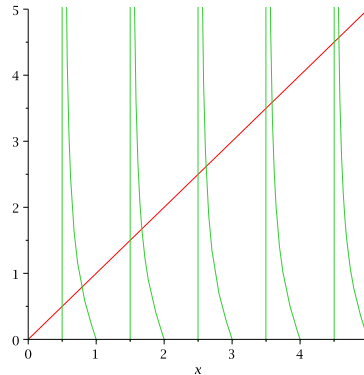
Similarly, the functions $\cos \alpha x$ with $\alpha = n\pi/L$ (including $n = 0$ in this case) correspond to the boundary condition $y'(0) = 0; \, y'(L) = 0$.

2. Eigenfunctions

There is something more general than Fourier series involved here. The boundary condition at each endpoints is allowed to be the requirement that some **fixed linear combination** of y and y' be zero at that point. For example, the boundary value problem

$$\frac{d^2 y}{dx^2} + \lambda y = 0; \, y(0) = 0; \, y(L) + y'(L) = 0 \tag{1}$$

has a non-trivial solution of the form $\sin \alpha x$ when $\sin \alpha L + \alpha \cos \alpha L = 0$. There are still infinitely many such α, but they are characterized by $\alpha = -\tan \alpha L$ — an equation that has one root in each interval of

the form $[(k - \frac{1}{2})\pi/L, (k + \frac{1}{2})\pi/L]$. As before, $\lambda = \alpha^2$, but there is no simple expression for α. These functions are of the form $\sin \alpha x$ because we required that $y(0) = 0$; a different boundary condition at $x = 0$ would change the form of the function. It can be shown by the method used in the first example that only these positive values of λ allow nonzero solutions of the boundary value problem. To take a specific example, take $L = \pi$, so the roots are in intervals of length 1. Here is a graph of $y = -\tan \pi x$ and $y = x$.
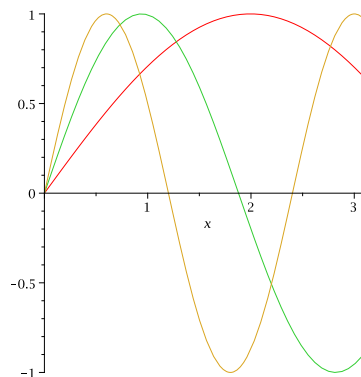

The vertical lines are the **asymptotes** of $y = -\tan \pi x$ at $(k - \frac{1}{2})\pi/L$ and appear as a **feature** of Maple's graphing engine. The first five roots of $x = -\tan \pi x$ computed by Maple are:

$$0.7876372942, 1.671605625, 2.6162135853.586552746, 4.568591746.$$

The corresponding values of λ are the squares of these numbers:

$$0.6203725072, 2.794265366, 6.844573522, 12.86336060, 20.87203054.$$

The graphs of the first three $\sin \alpha x$ are shown below.

This time, the eigenfunctions do not have a common period, so there is no clear interpretation of the series outside $[0, L]$, even if the series can be shown to converge.

In these problems, we are considering the effect of the **linear operator** d^2/dx^2 on the **linear space** of functions defined on the interval $[0, L]$ satisfying certain **homogeneous conditions** at the **boundary points** $x = 0$ and $x = L$. We identified functions taken into **multiples of themselves** by the operator. In **vector spaces**, such objects were called **eigenvectors** with the multiplier being called an **eigenvalue**, and in these **function spaces** the usual name is **eigenfunction**. In finite dimensional vector spaces, operators were characterized (in most cases) by their eigenvalues and eigenvectors, so that the behavior at a general vector could be found from the eigenvalues and eigenvectors. We aim for a corresponding result in function spaces.

The general theory predicts that the eigenfunctions will be **orthogonal** on $[0, L]$. This allows an arbitrary function $f(x)$ to be expressed as an **infinite** linear combination of eigenfunctions with coefficients

determined by integrating the product of $f(x)$ with the eigenfunctions. In the case where the eigenfunction expansion reduces to a variation on the Fourier series, the norms of the eigenfunctions are the **same** as in the Fourier series, so the **same formulas** may be used. It is worth noting that $f(x)$ need not satisfy the boundary conditions used to define the eigenfunctions. As in the case of Fourier series, the eigenfunction expansion will represent an extension of the function with a singularity at the end of defining interval. In particular, we can take $f(x) = 1$ and produce an expansion for the eigenfunctions determined by (1) with $L = \pi$.

The details are worth showing although you will not be required to do hand computation with the eigenfunctions defined by (1). Orthogonality requires the following evaluation with $m \neq n$.

$$\int_0^\pi \sin\alpha_m x \sin\alpha_n x \, dx = \frac{1}{2}\int_0^\pi \cos(\alpha_m x - \alpha_n x) - \cos(\alpha_m x + \alpha_n x) \, dx$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha_m x - \alpha_n x)}{\alpha_m - \alpha_n} - \frac{\sin(\alpha_m x + \alpha_n x)}{\alpha_m + \alpha_n}\right]_0^\pi$$

$$= \frac{1}{2}\left[\frac{\sin(\alpha_m \pi - \alpha_n \pi)}{\alpha_m - \alpha_n} - \frac{\sin(\alpha_m \pi + \alpha_n \pi)}{\alpha_m + \alpha_n}\right]_0^\pi$$

To complete the evaluation, we replace α_m by $-\tan\alpha_m\pi$ and α_n by $-\tan\alpha_n\pi$ **in the denominator**, and use the formula for the sine of a sum or difference in the numerator. This reduces the bracketed expression to $(\cos\alpha_m\pi)(\cos\alpha_n\pi) - (\cos\alpha_m\pi)(\cos\alpha_n\pi) = 0$, as expected. If $m = n$, the calculation is similar except that $\cos(\alpha_m x - \alpha_n x) = 1$, so its integral is π. The other term integrates to

$$\frac{\sin 2\alpha\pi}{2\alpha} = \frac{\sin\alpha\pi}{\alpha}\cos\alpha\pi = \cos^2\alpha\pi = \frac{1}{1 + \tan^2\alpha\pi} = \frac{1}{1 + \alpha^2}.$$
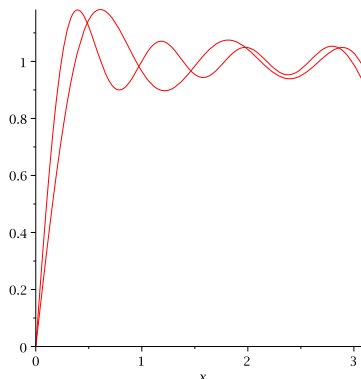
This leads to the coefficient of $\sin\alpha x$ being

$$\frac{2(1 + \alpha^2)}{1 + \pi(1 + \alpha^2)}\int_0^\pi f(x)\sin\alpha x \, dx$$

When $f(x) = 1$, this reduces to

$$\frac{2(1 + \alpha^2)}{1 + \pi(1 + \alpha^2)}\frac{1 - \cos\alpha\pi}{\alpha}.$$

The general proof of orthogonality is hidden in this calculation; we have just found a different description of a function whose derivative is the quantity integrated when testing orthogonality.

These coefficients were used to graph the sums of the first five terms and the first eight terms of the eigenfunction expansion of $f(x) = 1$. Here are the results.

Note that the left endpoint has $y(0) = 0$ for both curves, but the right endpoint has different values of $y(\pi)$, although $y(\pi) + y'(\pi) = 0$ for both curves. This is expected for this case since each term of the eigenfunction expansion has $y(\pi) \neq 0$

These first six coefficients were found to be

$$1.206263840, 0.1709830454, 0.3173400705, 0.1269152395, 0.1667164509, 0.09334414723.$$

Computers allow a systematic treatment of these more general problems, and you will use this resource in applications. However, homework and exams will emphasize problems whose eigenfunctions resemble Fourier series because they are more familiar and often lead to formulas for the solution.

The properties of the problem that allow a systematic study of its solution are that the **second derivative operator** is linear on the space of all smooth functions on $[0, L]$. The differential equation asserts that this operator takes y to a multiple of itself. As noted earlier in this section, such multiples (which are λ for the negative of the second derivative operator) are called **eigenvalues** of the operator, and when considering operators on spaces of functions, we refer to **eigenfunctions**.

3. Self-adjoint operators

In a finite dimensional space, the eigenvalues and eigenvectors of an operator given by a **symmetric matrix** have special properties. All eigenvalues are **real numbers** and there is an **orthogonal basis** of eigenvectors. The analog for differential operators on a space of functions on an interval I is an equation in **self-adjoint form**

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + (q(x) + \lambda p(x))y = 0$$

where $p(x)$ and $r(x)$ are **positive** on I. The presence of the function $p(x)$ gives a generalization of the usual eigenvalue problem, and the eigenfunctions will be orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_I p(x)f(x)g(x)\,dx.$$

The proof of Theorem 12.3(d) in the text shows how one works with this more general setting.

In this expression, $p(x)$ and $r(x)$ play special roles: $p(x)$ is the weight function for the inner product; the roots of $r(x)$ are points that may be used as endpoints with **no additional restriction** on the values of y and y'.

4. Integrating factors

Although the self-adjoint form looks special, an **arbitrary** second order operator can be put in that form simply by multiplying by a suitable function that allows the terms of order one and two to have the required form. As in other examples of this methods, this function is called an **integrating factor**. Indeed, the method of discovery and the expression for this integrating factor are similar to case of the first order linear equation. Multiplying $y'' + b(x)y'$ by $r(x)$ gives the derivative of $r(x)y'$ precisely when $r' = br$, so $b(x)$ must be the derivative of $\ln r(x)$. Thus r is given by integrating $b(x)$ and exponentiating the result. On any interval where this can be done, e.g., an interval where $b(x)$ is continuous, the result is **positive**, as required by the general theory.

A similar characterization of the integrating factor can be given starting from **any** expression $a(x)y'' + b(x)y'$; it is not necessary to divide by $a(x)$ as a separate step. If the integrating factor is denoted $\mu(x)$, then $a(x)\mu(x) = r(x)$ and $b(x)\mu(x) = r'(x)$. The equation $\big(a(x)\mu(x)\big)' = b(x)\mu(x)$ simplifies to first order linear homogeneous equation for $\mu(x)$. Such equations are separable, so there is a standard solution that is easy to derive. Nothing is gained by writing the solution as a formula. Indeed, one **loses** the ability to see a shortcut to the solution when following the standard method of solution. Once $\mu(x)$ has been found, $r(x) = a(x)\mu(x)$, and the self-adjoint form becomes visible after the original equation is multiplied by $\mu(x)$.

5. The Sturm-Liouville Theory

The orthogonality of eigenfunctions with respect to the positive weight function $p(x)$ proved in Theorem 12.3(d) allows endpoints where $r(x) = 0$ as well as those where a boundary condition must be prescribed. This allows examples for which the equation is **singular** at one or both boundary points and the solution is required to satisfy a suitable condition at that point that limits attention to a one-dimensional space of functions. The text gives examples arising from Bessel's equation and Legendre's equation. The nature of the boundary condition leads to simplicity of eigenvalues noted in Theorem 12.3(b). The linear independence of eigenfunctions for different eigenvalues claimed by Theorem 12.3(c) is a general property that has an easy proof based of a trick: one supposes that a **simplest** dependence relation has been found; the operator is applied to it and the result simplified using the fact that the terms are eigenfunctions; these two dependence relations are then combined to get a simpler non-trivial relation. A deeper study is required for part (a) of the theorem. Such a study can be found in Hans Sagan, "Boundary and Eigenvalue Problems in Mathematical Physics", Dover Publications, NY (my copy has a price of $17.95 on the cover — Dover aims to publish inexpensive paperback books, many of which were textbooks abandoned by their original publisher). Although a general proof may be difficult, the verification of the properties is easy in any particular case.

It is conventional to let the eigenvalue be the λ in the equation, so the operator is the **negative** of the sum of the other terms, i.e.,

$$-\left(\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + q(x)y\right)$$

This hides the quantity $p(x)$, but this is the weight function, so it will appear in the orthogonality relation. Indeed, this weight function should **not** be hidden since it identifies the problem as an extended form of the eigenvalue problem that is considered for symmetric matrices.

6. The parametric Bessel equation

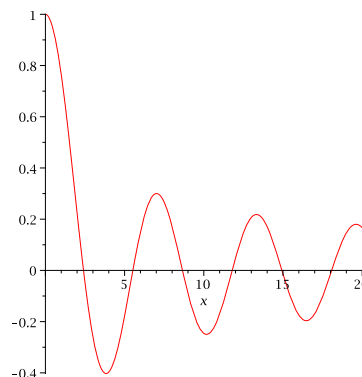The Bessel function of the first kind of order $\nu \geq 0$, denoted $J_\nu(x)$, satisfies

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0$$

with

$$\lim_{x\to 0} x^{-\nu}J_\nu(x)$$

equal to a particular finite, nonzero value. The **method of Frobenius** produces a **series solution** with this property. See sections 5.6 through 5.8 of the Math 244 text by Boyce and DiPrima or sections 5.2 and 5.3 of the present textbook for details. A family of eigenfunctions can be found for any fixed nonnegative real value of ν, but the case $\nu = 0$ is a good illustration of general properties of these functions.

The singular point at zero is a natural location for one of the boundaries. To select another boundary, we note that $J_\nu(x)$ is an oscillatory function, as illustrated by the following graph of $J_0(x)$

Maple can also produce the list of roots of $J_\nu(x) = 0$. For $\nu = 0$, the first six values (those visible on our graph) are

$$2.404825558, 5.520078110, 8.653727913, 11.79153444, 14.93091771, 18.07106397.$$

Setting α equal to the successive solutions of $J_\nu(x) = 0$ allow $x = 1$ to be a boundary at which $J_\nu(\alpha x) = 0$. Thus, the functions $J_\nu(\alpha x) = 0$ with fixed ν and boundary conditions requiring that

$$\lim_{x \to 0} x^{-\nu} f(x) \text{ is finite,}$$

and $f(1) = 0$ leads to the sequence of functions $J_\nu(\alpha_n x)$ with fixed ν and a sequence of α_n determined by the zeros of $J_\nu(x)$.

It remains to find the differential equation satisfied by the $J_\nu(\alpha x)$, to use that equation to characterize these as eigenfunctions of some operator, and to show that these functions form a **complete** basis for functions satisfying the boundary conditions.

We find an equation satisfied by this sequence of functions and put it in **self-adjoint form**, but omit the proof of completeness..

Writing $u = \alpha x$ and $w = J_\nu(u)$, we have

$$\frac{du}{dx} = \alpha \quad \text{and} \quad u^2 \frac{d^2 w}{du^2} + u \frac{dw}{du} + (u^2 - \nu^2)w = 0$$

Now,

$$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx} = \alpha \frac{dw}{du}$$

and

$$x \frac{dw}{dx} = \alpha x \frac{dw}{du} = u \frac{dw}{du}.$$

Similarly,

$$x^2 \frac{d^2 w}{dx^2} = u^2 \frac{d^2 w}{du^2}.$$

Thus,

$$x^2 \frac{d^2 w}{dx^2} + x \frac{dw}{dx} + (\alpha^2 x^2 - \nu^2)w = 0$$

This equation is known as **the parametric Bessel equation**.

The integrating factor to put this in self-adjoint form is $1/x$, so that form of the equation is

$$\frac{d}{dx}\left(x \frac{dw}{dx}\right) + (\alpha^2 x - \frac{\nu^2}{x})w = 0,$$

and the eigenfunctions will be orthogonal on $[0, 1]$ **with respect to the weight** x. That is,

$$\int_0^1 x J_\nu(\alpha_m x) J_\nu(\alpha_n x)\, dx = 0$$

if $m \neq n$. The eigenvalue is $\lambda = \alpha^2$.

As an example, the series for $1 - x^2$ on the interval $[0, 1]$ with $f(1) = 0$ was computed. Five terms were sufficient for the graph of the series to be visually coincide with the graph of the original function. The first five coefficients, computed by Maple, are:

$$1.108022262, -0.1397775052, 0.04547647069, -0.02099090194, 0.01163624313$$

After this, the remaining coefficients are less than 10^{-2} in absolute value (only 12 coefficients exceed 10^{-3} in absolute value; only 32 exceed 10^{-4} in absolute value; 80 exceed 10^{-5} in absolute value). These computations used only the definition of the coefficients as a quotient of integrals: Maple is **allowed to** use any sufficiently accurate method to evaluate the integrals, but it is not **required to** use the properties of Bessel functions used in Definition 12.8 of the textbook. Sophisticated systems like Maple may use slow numerical methods to save the programmer's time.

The examples of the Fourier-Bessel expansion of x on $[0, 3]$ using Bessel functions of order one given in the text show a difference in Figure 12.21(a) with five terms because the boundary condition at $x = 3$ satisfied by all eigenfunctions is not satisfied by the given function. The graph in Figure 12.21(b) shows the **extension** of the sum of the first ten terms of the series to a **larger interval**. The solution outside the given interval is of questionable significance, but it does illustrate that the extension introduces a singularity into the extension of the given function.

7. Legendre polynomials

The Legendre polynomials satisfy an equation whose self-adjoint form is

$$\frac{d}{dx}\left((1 - x^2)\frac{dy}{dx}\right) + n(n + 1)y = 0$$

General properties of series solutions about $x = 0$ show that one solution is a polynomial of degree n. Furthermore $x = \pm 1$ are singular points of the equation, and the **method of Frobenius** shows that these polynomials are the only solutions bounded a these singular points, and they are all nonzero at the boundary points. Using the Legendre polynomials $P_n(x)$ as eigenfunctions with eigenvalue $n(n + 1)$ gives a family of orthogonal functions on $[-1, 1]$. Because we are working with an interval both of whose endpoints are singular, the only boundary conditions are that the functions be **bounded** at the endpoints. Note that the $P_n(x)$ with n even contain only even powers of x, so are even functions, and those with n odd contain only powers of x, so are odd functions.

8. Other orthogonal polynomials

Exercises introduce Laguerre's equation

$$x\frac{d^2y}{dx^2} + (1 - x)\frac{dy}{dx} + ny = 0,$$

and Hermite's equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0.$$

Each of these has a polynomial solution of degree n and may be put in self-adjoint form. We are interested in these equations for all nonnegative integer values of n and the coefficient of y in these equations is the eigenvalue. The weight function $p(x)$ is both the integrating factor (because the zero order term in the original equation has the form λy), and term that identifies any special points that may be taken as boundary points with no additional boundary conditions (because the second order term is y''). Because $p(x)$ decreases rapidly at $x = +\infty$ for Laguerre's equation and at both $\pm\infty$ for Hermite's equation, these points are available as boundary points at which no additional assumptions are required. This hint should have been provided with the exercises dealing with these polynomials since the requirement of explicit boundary conditions at other boundary points was not given sufficient emphasis. The significance of these examples is that they define a **natural** family of orthogonal polynomials for their weight functions.

9. How will this be used?

Eigenfunction expansions generalize Fourier series and are determined using similar methods. Indeed, some are variants on Fourier series corresponding to particular boundary conditions. However, this is not reason enough to introduce them.

The power of eigenfunction expansions lies in their ability to represent all reasonable functions satisfying certain boundary conditions and to **have a predictable behavior when differentiated**. This will justify the **method of separation of variables** for solving certain **partial differential equations**. The theory of partial differential equations emphasizes boundary value problems since there are only a limited class of physical situations allowing anything like the initial value problems that were used to organize the study of ordinary differential equations. A good beginning to this study is a consideration of the role of boundary data in ordinary differential equations.

Initially, we will consider problems on **rectangular regions in the plane**, and we will give complete solutions to some classical equations in that context. In order to apply our methods to other regions, the problems will need to be described in a way that is independent of coordinates, and then introduce a special coordinate system to that suits region on which we want to solve the equation. The parametric Bessel functions and Legendre polynomials will play a role in the study of important regions in two and three dimensions.

End of supplement