Mathematics 452 Abstract Algebra II Spring 2015

PRACTICE FINAL EXAMINATION

No books or cellphones
Calculators may NOT be used
Show and explain all of your work

You should know the definitions of the mathematical objects we studied this semester from chapters 3,4,11,12,14,15 of Artin:

Field, vector space, basis,dimension, linear independence, span characteristic polynomial, invariant subspace, eigenspace, ring, ideal, homomorphism, quotient ring, Gaussian integers, algebraic integer/number, modules, presentations, cyclic degree, isomorphism, prime element, irreducible element, PID, UFD, Noetherian ring maximal, prime ideals, algebraic, degree, Galois group

You should know the statements and uses of major theorems that we covered this semester:

Euclidean algorithm, Jordan Form, Dimension(or Rank-Nullity) Formula, Structure of modules over Euclidean rings, Theory of Finite Fields, Nullstellensatz, Isomorphism theorems, Correspondence Theorem, Chinese remainder Theorem, Gauss Lemma, R UFD then R[x] is, Rational canonical form, Fundamental theorem of algebra

The final exam on May 8 will be similar to this review exam. Some questions on this exam may appear on the final exam.

1. A complex matrix A is called Hermitian if the transpose of A equals the complex conjugate of A.

   a) Show the the set of Hermitian $n \times n$ matrices form a real vector space W_n.
   b) Find the dimension of W_n and a basis for W_n.

2. Let A be an $m \times n$ complex matrix and let B be an $n \times m$ complex matrix.

   a) Show that any nonzero eigenvalue λ of AB is an eigenvalue of BA.

   b) Give an example of matrices A, B as above such that 0 is an eigenvalue of AB but not of BA.

   c) Let I_m be the $m \times m$ identity martix. Show that $I_m - AB$ is invertible if and only if $I_n - BA$ is invertible.

3. Let V_n be the real vector space of polynomials of degree at most n. Let $D : V_n \to V_n$ be the derivative map.

a) Let $L : V_n \to V_n$ be the map $L(p) = D^2(p) - p$. Show that the polynomials of degree at most m form an invariant subspace for L.

b) Find a nonzero invariant subspace of V_n which does not consist of all polynomials in V_n with degree at most m for some m.

c) Determine all invariant subspaces of L

4. The complex matrix A has characteristic polynomial $t(t-1)^3(t^2+t+1)$. Determine a set of matrices such that any such A is similar to exactly one member of your set.

5. Show that every square complex matrix of finite order is diagonalizable. Find a finite order 2 x 2 matrix over the field with 2 elements which is not diagonalizable

6. Suppose that A is a complex 6×6 matrix such that $A^2(A-2I)^3 = 0$ but no nonconstant polynomial in A of degree less than 5 equals 0.

a) What are the eigenvalues of A?

b) What are the possible Jordan forms for A?

7. Let R be a commutative integral domain.

a) Define what it means for an element $r \in R$ to be irreducible.
b) Define what it means for an element $r \in R$ to be prime.
c) Show that every prime element is irreducible.
d) Give an example of a domain R and a prime element in R that is not irreducible.

8. Factor the Gaussian integer $1 + 5i$ as a product of irreducible Gaussian integers.

9. Let R be a finite ring.

a) Show that if R is a domain then it is a field.
b) Does there exist a domain R with 12 elements? Why or why not?

10. Show that an ideal I in a ring R is a free R module if and only if I is a principal ideal (t) for some $t \in R$ which is not a zero divisor.

11. Find the abelian group presented by

$$\begin{pmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 2 & 2 \end{pmatrix}$$

12. Up to isomorphism how many abelian groups of order $3^4 7^2 11^4$ are there?

13. Does the field $Q(\sqrt{-2})$ contain a square root of -1?

14. Let a be an integer which is not a square. Prove that the degree of $\mathbf{Q}(a^{1/4})$ over the rational field is 4.

15. Let $F = F_2$ and let K be a field with 16 elements.

a) How many generators of the multiplicative group of K are there?

b) How many elements $\alpha \in K$ satisfy $K = F(\alpha)$?

16. How many subfields does a field with 2^{10} elements have? Classify them.

17. Let F be a finite field with $q = p^r$ elements for some prime p. Let $L(z) = z^p$.
    a) show that L is a linear map of the F_p vector space F to itself.
    b) When $q = 3^2$ determine the structure of F as a module over $F_p[x]$, when $xv = Lv$.

18. True and False: Indicate whether the following statements are true or false. If false provide a counterexample. If true explain why.

    1. Every module over a Euclidean domain is isomorphic to a direct sum of cyclic modules.
    2. The rings $(\mathbf{Z}/10\mathbf{Z})[x]$ and $\mathbf{Z}[x,y]/(10y)$ are isomorphic rings.
    3. An abelian group presented by a square matrix of integers with nonzero determinant is a finite group.
    4. There are at no more than 10 irreducible monic polynomials of degree 6 in the ring $\mathbf{Z}/2\mathbf{Z}[x]$.