MATH 495, Fall '06

A brief, non-rigorous introduction to Itô calculus for Brownian motion.

1. Preliminaries

A. Limits of random variables.

We shall need to make statements of the form

$$\lim_{n\to\infty} X_n = Y \qquad (1)$$

where $Y, X_1, X_2, \dots$ are all random variables. There are many possible ways to interpret (1).

a) $X_n \xrightarrow{a.s.} Y$ as $n\to\infty$ if

$$P\left(\lim_{n\to\infty} X_n = Y\right) = 1$$

Here "a.s." stands for almost-surely and $X_n \xrightarrow{a.s} Y$ as $n\to\infty$ is referred to in words as $\{X_n\}$ converges to Y almost surely, or $\{X_n\}\to Y$ with probability one.

a) seems the most natural definition, but there is another definition that is more convenient for us.

b) "$X_n \xrightarrow{m.s} Y$ as $n\to\infty$" or "$\{X_n\}$ converges to Y in mean square" if

$$\lim_{n\to\infty} E\left[|X_n - Y|^2\right] = 0 \qquad (2)$$

These two notions of convergence are related but not equivalent. We shall actually not worry about proving limits in either sense, but we need these definitions in order to make mathematically coherent statements about Itô calculus.

2. Taylor polynomials

a) Let f be a twice continuously differentiable function defined on an interval about the point a. The Taylor polynomial of order 2 for f at a is

$$p_2(x) = f(a) + f'(a)\cdot(x-a) + \tfrac{1}{2} f''(a)(x-a)^2 \qquad (3)$$

p_2 should be thought of as the "best" approximation to f by a quadratic function in a small interval about a. It is best in the sense that it is the unique quadratic function satisfying $p_2(a) = f(a)$, $p_2'(a) = f'(a)$, $p_2''(a) = f''(a)$ and the unique quadratic function such that $\lim_{x\to a} \frac{f(x)-p_2(x)}{(x-a)^2} = 0$.

Example $f(x) = e^x$. They order 2 Taylor polynomial at a of f is $p_2(x) = e^a + e^a(x-a) + \frac{1}{2}e^a(x-a)^2$

b) The idea of Taylor polynomial approximation can be extended to multi-variable function.

Let $g(t,x)$ be twice continuously differentiable. The Taylor polynomial of order 2 for g at (t_0, a) is

$$P_2(t,x) = g(t_0,a) + \frac{\partial g}{\partial t}(t_0,a)(t-t_0) + \frac{\partial g}{\partial x}(t_0,a)(x-a)$$
$$+ \frac{1}{2}\frac{\partial^2}{\partial x^2} g(t_0,a)(x-a)^2 + \frac{\partial^2}{\partial x \partial t} g(t_0,a)(t-t_0)(x-a)$$
$$+ \frac{1}{2}\frac{\partial^2}{\partial t^2} g(t_0,a)(t-t_0)^2 \qquad (4)$$

$P_2(t,x)$ is the unique polynomial in (t,x) of order 2 such that the values of $P_2(t,x)$ and its first and second derivatives match those of g at (t_0,a):

$$P_2(t_0,a) = g(t_0,a) \qquad \frac{\partial P_2}{\partial t}(t_0,a) = \frac{\partial g}{\partial t}(t_0,a)$$
$$\frac{\partial P_2}{\partial x}(t_0,a) = \frac{\partial g}{\partial x}(t_0,a)$$
$$\frac{\partial^2 P_2}{\partial x^2}(t_0,a) = \frac{\partial^2}{\partial x^2} g(t_0,a), \quad \frac{\partial^2 P_2}{\partial t \partial x}(t_0,a) = \frac{\partial^2}{\partial t \partial x} g(t_0,a), \quad \frac{\partial^2 P_2}{\partial t^2}(t_0,a) = \frac{\partial^2 g}{\partial t^2}(t_0,a)$$

The error made in approximating $g(t,x)$ by $P_2(t,x)$ will be small relative to the square of the distance from (t,x) to (t_0,a).

<u>Example</u> $g(t,x) = e^{tx+t}$ $\qquad t_0 = 1, \; a = 1$

$$P_2(t,x) = e^2 + 2e^2(t-1) + e^2(x-1)$$
$$+ \frac{e^2}{2}(x-1)^2 + 3e^2(x-1)(t-1) + 2e^2(t-1)^2$$

c) <u>Quadratic variation of Brownian motion</u>

We have done this before. We repeat the basic fact.

Partition $[0,t]$ into n subintervals by $t_0 = 0, t_1 = \frac{t}{n}, \ldots, t_n = n\frac{t}{n} = t$. Then

$$\sum_{i=0}^{n-1} \left(B_{(i+1)\frac{t}{n}} - B_{i\frac{t}{n}} \right)^2 \xrightarrow[n\to\infty]{m.s.} t$$

2. Stochastic Integrals

Let $\{Y_t, t \geq 0\}$ be a random process in continuous time. Let $\{\phi_t, t \geq 0\}$ be a second process.
Let

$$t_0 = 0, \; t_1 = \frac{t}{n}, \; t_2 = \frac{2t}{n}, \; \dots, \; t_n = \frac{nt}{n} = t$$

be a partition of $[0,t]$ into n subintervals.
Consider the sum

$$\sum_{i=0}^{n-1} \phi_{t_i} \left(Y_{t_{i+1}} - Y_{t_i} \right) \qquad (5)$$

Interpretation. Suppose at each time t_i we are allowed to place a bet on the movement of Y over the time interval $[t_i, t_{i+1}]$, such that if we bet \$$b$, we earn $b(Y_{t_{i+1}} - Y_{t_i})$ (of course this is a loss of $b(Y_{t_{i+1}} - Y_{t_i})$). Then (5) represents our total earnings over $[0,t]$ if we employ the strategy that bets ϕ_{t_i} on the movement of Y on $[t_i, t_{i+1}]$.

Example 1 Let $S_t = S_0 \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma B_t\}$, where B is a Brownian motion be the risk-neutral Black-Scholes stock price. At each t_i we rebalance our portfolio to hold ϕ_{t_i} shares of stock. The profit or loss from this investment over time interval $[t_i, t_{i+1}]$ is in dollar amount $\phi_{t_i}[S_{t_{i+1}} - S_{t_i}]$ so

$$\sum_{i=0}^{n-1} \phi_{t_i} [S_{t_{i+1}} - S_{t_i}] \qquad (6)$$

is the total amount we earn from the investment strategy specified by $\{\phi_t\}$.

Suppose we start with an initial amount $\Pi_0(x) = x$ of money. At each t_i we invest in ϕ_{t_i} shares of stock over $[t_i, t_{i+1}]$ while investing the rest of our money at the risk-free rate r. What are our total earnings? Let Π_{t_i} be the value of the portfolio at t_i. Then the portfolio over $[t_i, t_{i+1}]$ holds ϕ_{t_i} shares of stock with $\Pi_{t_i} - \phi_{t_i} S_{t_i}$ invested at rate r. Hence

$$\Pi_{t_{i+1}} = \left(\Pi_{t_i} - \phi_{t_i} S_{t_i}\right) e^{r(t_{i+1} - t_i)} + \phi_{t_i} S_{t_{i+1}}$$

So

$$\Pi_{t_{i+1}} - \Pi_{t_i} = \left(\Pi_{t_i} - \phi_{t_i} S_{t_i}\right)\left(e^{r(t_{i+1} - t_i)} - 1\right) + \phi_{t_i}\left(S_{t_{i+1}} - S_{t_i}\right) \qquad (7)$$

and

$$\begin{aligned} \Pi_t - \Pi_0 &= \left(\Pi_{t_1} - \Pi_0\right) + \left(\Pi_{t_2} - \Pi_{t_1}\right) + \dots + \left(\Pi_t - \Pi_{t_{n-1}}\right) \\ &= \sum_{i=0}^{n-1} \left(\Pi_{t_i} - \phi_{t_i} S_{t_i}\right)\left(e^{r(t_{i+1} - t_i)} - 1\right) \\ &\quad + \sum_{i=0}^{n-1} \phi_{t_i}\left(S_{t_{i+1}} - S_{t_i}\right) \end{aligned} \qquad (8)$$

Example 2 Trading on a Brownian motion using the Brownian motion! Let $\{Y_t\} = \{B_t\}$ where B is a Brownian motion and let $\{\phi_t\} = \{B_t\}$ as well. Consider

$$\sum_{i=0}^{n-1} 2B_{t_i}(B_{t_{i+1}} - B_{t_i})$$

Note that

$$B_{t_{i+1}}^2 - B_{t_i}^2 - 2B_{t_i}(B_{t_{i+1}} - B_{t_i})$$

$$= B_{t_{i+1}}^2 - 2B_{t_i}B_{t_{i+1}} + B_{t_i}^2$$

$$= (B_{t_{i+1}} - B_{t_i})^2$$

Thus

$$\sum_{i=0}^{n-1} 2B_{t_i}(B_{t_{i+1}} - B_{t_i}) = \sum_{i=0}^{n-1}(B_{t_{i+1}}^2 - B_{t_i}^2) - \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2$$

$$= B_t^2 - B_0^2 - \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2$$

$$= B_t^2 - \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2 \qquad (9)$$

as $B_0 = 0$.

We now want to consider what happens as $n \to \infty$. In the betting interpretation we are allowing more frequent betting over smaller and smaller time intervals as n increases. We define the stochastic integral

$$\int_0^t \phi_s \, dY_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} \phi_{t_i}(Y_{t_{i+1}} - Y_{t_i})$$

if this limit exists.

Of course we have to specify in which sense the limit has to be taken. For our purposes we can make a more precise definition as follows:

$X_t = \int_0^t \phi_s dY_s$ is that random variable, if it exists, such that $\sum_{i=0}^{n-1} \phi_{t_i}(Y_{t_{i+1}} - Y_{t_i}) \xrightarrow[n\to\infty]{m.s} X_t$.

Example. We have the remarkable formula

$$\int_0^t 2B_s \, dB_s = B_t^2 - t$$

By equation (9) and by the fact about the quadratic variation of $\{B_t\}$ --- see page 4 --

$$\int_0^t 2B_s \, dB_s = \lim_{n\to\infty} \sum_{i=0}^{n-1} 2B_{t_i}(B_{t_{i+1}} - B_{t_i}) =$$

$$B_t^2 - \lim_{n\to\infty} \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_t)^2 = B_t^2 - t$$

where the limit is in the mean square sense.

Here is a non-rigorous statement of a very basic fact. It states that stochastic integrals of the form $\int_0^t \phi_s \, dB_s$ are well-defined for a broad class of $\{\phi_t\}$ if B is a Brownian motion.

'Theorem'. Assume that with probability 1 the paths

(a) With probability 1 the paths of ϕ are piecewise continuous

(b) ~~(Adaptness)~~ For every $0 < s < t$, $B_t - B_s$ is independent of $\{\phi_u, u \le s\}$

(c) $E\left[\int_0^T \phi_s^2 \, ds\right] < \infty$

Then

(i) $\int_0^t \phi_s \, dB_s$ is defined for $t \le T$

(ii) $\int_0^t \phi_s \, dB_s$, $t \le T$ is a martingale

(if $0 < s < t \le T$
$$E\left[\int_0^t \phi_u \, dB_u \,\Big/\, B_\tau, \phi_\tau, \tau \le s\right] = \int_0^s \phi_u \, dB_u \Big)$$

(iii) For every t
$$E \int_0^t \phi_s \, dB_s = 0, \quad E\left[\left(\int_0^t \phi_s \, dB_s\right)^2\right] = \int_0^t \phi_s^2 \, ds$$

Conditions (a) and (c) will always be true in our applications so we will generally not check them.

Condition (b) in words says that ϕ does not anticipate B -- that is knowledge of the past of ϕ up to time s has no bearing on the behavior of $B_t - B_s$. This condition of non-anticipation is very natural in the finance context. If we are betting on fluctuations in the future of B, we are not allowed to anticipate what those fluctuations might be. (Otherwise we could make unlimited amounts of money!)

Consequence (ii), that $\int_0^t \phi_s \, dB_s$, $t \le T$, is a martingale is very natural. The integral represents the gain on continuous betting on the future increments of B. As B is a martingale and as ϕ does not anticipate B, we cannot expect to gain or lose on average and so $\int_0^t \phi_s \, dB_s$ is also a martingale.

Definition of notation

$$dX_t = \alpha_t \, dt + \beta_t \, dY_t \quad \text{means}$$
$$X_t - X_0 = \int_0^t \alpha_s \, ds + \int_0^t \beta_s \, dY_s .$$

Formally and heuristically we think of dX_t as $dX_t = X_{t+dt} - X_t$ and $dX_t = \alpha_t dt$ as $X_{t+dt} - X_t = \alpha_t \, dt + \beta_t (Y_{t+dt} - Y_t)$

<u>Example</u> From the previous example

$$2B_t\,dB_t = d(B_t^2) - dt$$

or

$$d(B_t^2) = 2B_t\,dB_t + dt \qquad (10)$$

<u>Important and amusing formal consequence of (10)</u>

$$\boxed{(dB_t)^2 = dt \qquad (11)}$$

!

Why? From (10)

$$B_{t+dt}^2 - B_t^2 = 2B_t(B_{t+dt} - B_t) + dt$$

~~Ad~~. Subtract $2B_t(B_{t+dt} - B_t)$ from both sides

$$B_{t+dt}^2 - 2B_t B_{t+dt} + B_t^2 = dt$$

or

$$(B_{t+dt} - B_t)^2 = dt$$

or

$$(dB_t)^2 = dt$$

<u>Another example important for finance.</u>

We return to the portfolio example in which ϕ_t is the number of shares of stock to hold over interval $[t, t+dt]$. Recall from (8)

$$\Pi_t - \Pi_0 = \sum_{i=0}^{n-1} (\Pi_{t_i} - \phi_{t_i} S_{t_i})(e^{r(t_{i+1}-t_i)} - 1)$$
$$+ \sum_{i=0}^{n-1} \phi_{t_i}(S_{t_{i+1}} - S_{t_i}) \qquad (8)$$

Now $t_{i+1} - t_i = t/n$ and for n large $t_{i+1} - t_i$ is small and

$$e^{r(t_{i+1}-t_i)} - 1 \approx 1 + r(t_{i+1}-t_i) - 1 = r(t_{i+1}-t_i)$$

The error is of order $(t_{i+1}-t_i)^2 = t^2/n^2$ and so is negligeable.

Using this approximation, the first term in (8) is (approximately) the Riemann sum

$$\sum_{i=0}^{n-1} (\Pi_{t_i} - \phi_{t_i} S_{t_i}) r(t_{i+1} - t_i)$$

and as $n \to \infty$ this converges to $\int_0^t (\Pi_u - \phi_u S_u) r\,du$.
Therefore taking $n \to \infty$ in (8)

$$\Pi_t - \Pi_0 = \int_0^t (\Pi_u - \phi_u S_u) r\,du + \int_0^t \phi_u\,dS_u$$

3) Itô's rule

<u>Motivation</u> The Black-Scholes price model is

$$S_t = S_0 \exp\{ (\mu - \tfrac{1}{2}\sigma^2) t + \sigma B_t \}$$

Can we express S_t as a stochastic integral? That is, are there α_t and β_t so that

$$dS_t = \alpha_t dt + \beta_t dB_t \quad ?$$

More generally we consider the problem:

what, if anything, is

$$d g(t, X_t)$$

if $\quad dX_t = \alpha_t dt + \beta_t dB_t \quad ?$

The answer to this question is Itô's rule; this prescribes the following procedure

1) Approximate $\quad dg(t, X_t) = g(t+dt, X_{t+dt}) - g(t, X_t)$ using a Taylor polynomial of order 2 at (t, X_t).

2) Where $(dB_t)^2$ appears in this approximation replace it by dt. Set terms containing $dB_t\, dt$ or $(dt)^2$ to zero.

<u>Example</u>

a) Let $f(x) = x^2$. Then $B_t^2 = f(B_t)$

$$df(B_t) = f(B_{t+dt}) - f(B_t) = f'(B_t)[B_{t+dt} - B_t] + \tfrac{1}{2} f''(B_t)\underbrace{(B_{t+dt} - B_t)^2}$$

$$= f'(B_t)\, dB_t + \tfrac{1}{2} f''(B_t)(dB_t)^2$$

Since $f'(x) = 2x$, and $f''(x) = 2$ we get

$$df(B_t) = 2B_t dB_t + (dB_t)^2 = 2B_t dB_t + dt$$

This recovers what we derived by direct calculation.

b) Let $X_t = (\mu - \tfrac{1}{2}\sigma^2)t + \sigma B_t$. Then the Black-Scholes price model is

$$S_t = g(X_t)$$

where $\quad g(x) = S_0 e^{x}$. Since $g'(x) = g''(x) = g(x)$

$$dg(X_t) = g(X_{t+dt}) - g(X_t)$$

$$= g'(X_t)(X_{t+dt} - X_t) + \tfrac{1}{2} g''(X_t)(X_{t+dt} - X_t)^2$$

$$= g(X_t)\, dX_t + \tfrac{1}{2} g(X_t)(dX_t)^2$$

Now, $\quad dX_t = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dB_t$. So, using $S_t = g(X_t)$,

$$dS_t = d\,g(X_t)$$

$$= S_t\left[(\mu - \tfrac{1}{2}\sigma^2)dt + \sigma\, dB_t\right] + \tfrac{1}{2}S_t\left[(\mu - \tfrac{1}{2}\sigma^2)dt + \sigma\, dB_t\right]^2$$

$$= S_t\left[(\mu - \tfrac{1}{2}\sigma^2)dt + \sigma\, dB_t\right] + \tfrac{1}{2}S_t\left[\sigma^2(dB_t)^2 + 2dB_t(\mu - \tfrac{1}{2}\sigma^2)dt + (\mu - \tfrac{1}{2}\sigma^2)(dt)^2\right]$$

Setting $(dB_t)^2$ to dt and ~~dB_t~~ eliminating terms with $dB_t dt$ ~~or~~ and $(dt)^2$ as specified by Professor Itô

$$dS_t = S_t\left[(\mu - \tfrac{1}{2}\sigma^2)\right]dt + \sigma S_t\, dB_t + \tfrac{1}{2}\sigma^2 S_t\, dt$$

$$= S_t \mu\, dt + \sigma S_t\, dB_t$$

↙ cancellation of $\frac{\sigma^2}{2}S_t dt$ terms!

or

$$\boxed{dS_t = \mu S_t\, dt + \sigma S_t\, dB_t \quad (12)\,!}$$

In terms of stochastic integrals

$$S_t = S_0 + \int_0^t \mu S_s\, ds + \int_0^t \sigma S_s\, dB_s.$$