640:495 Mathematical Finance: Notes, Portfolios and the Greeks.

This lecture is about portfolio analysis with the so-called Greeks. We shall be studying portfolios containing an asset and derivatives based on that asset. For definiteness, assume that the asset is stock in the fictitious XYZ Corp., but understand that it could be any risky asset. Throughout the lecture, the variable S will stand for the price of the stock. We shall be interested in studying how the value of a portfolio changes as S changes, and also as time or other model parameters change.

The following examples illustrate simple portfolios. In each case, we are concerned with expressing the portfolio value as a function of price and possibly other parameters, such as time, on which the portfolio value depends.

Examples.

(a) Consider a portfolio which just holds x shares of XYZ stock. If the price of the stock is S, its value is

$$L(s) = xS.$$

(b) Consider a porfoio which holds x shares of XYZ stock, y in cash, and z call options on the stock. We are interested in the value of the stock on one day, say $t = 0$, as a function of S. Suppose that the call price on this day is function $C(S)$ of the stock price. Then the value of the portfolio, as a function of S, is

$$L(S) = xS + y + zC(s).$$

(c) In example (b) we are taking a static view of the portfolio; we consider only its value at a fixed time, as a function of S. This time we will take a dynamic viewpoint, and at the same time consider general derivative securities. Consider a derivative security written on XYZ stock, which may be a put, a call, or whatever. We assume only that its price is given by a function $D(S, t)$, $S \geq 0$, $0 \leq t \leq T$; that is, $D(S, t)$ is the price of the option at time t if the stock price is S. Consider a portfolio holding x shares of stock and z derivatives D. Its value as a function of S and t is

$$L(S, t) = xS + zD(S, t).$$

This represents the portfolio value when x and z are held fixed.

NOTE WELL. Do not confuse $L(S, t)$ with $L(S_t, t)$ where $\{S_t; t \geq 0\}$ is a random price process. $L(S_t, t)$ gives the price of the portfolio as a function of time only along a random realization of the price process; $L(S, t)$ is a function of S and t specifying the portfolio value for any possible choice of S and t.

In this lecture, the Black-Scholes formula for a call expiring at T with strike X, when the volatility is σ and the risk-free interest is r, will be denoted,

$$C^{T,X}(S,t) = SN(d_1(S,t)) - e^{-r(T-t)}XN(d_2(S,t)),$$

where $d_1(s,t) = \dfrac{\ln(S/X)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, and $d_2(S,t) = d_1(S,t) - \sigma\sqrt{T-t}$. If we want to display the dependence on σ and r explicitly, we shall write $C^{T,X}(S,t;\sigma,r)$.

The corresponding Black-Scholes prices for puts will be denoted by $P^{T,X}(S,t)$.

Further examples:

(d) A portfolio is short 10 calls, priced according to Black-Scholes. $L(S,t,\sigma,r) = -10C^{T,X}(S,t;\sigma,r)$ expresses its value as a function of all the model parameters, the price, and time.

(e) A portfolio has 5 shares of stock and 3 puts, expiring at T with strike X, priced according to Black-Scholes. Its value is $L(S,t) = 5S + 3P^{T,X}(S,t)$. Here, σ and r are considered fixed and not included in L.

We will now define the Greeks of a portfolio. These are simply partial derivatives of the portfolio function L that express that rates of change of portfolio value with respect to the input variables. The most imporant is delta:

$$\Delta_L(S) = \frac{\partial L}{\partial S}(S).$$

When L depends on other variables, these are also indicated in the notation; thus

$$\Delta_L(S,t) = \frac{\partial L}{\partial S}(S,t).$$

The greek gamma is the second derivative in S.

$$\Gamma_L(S) = \frac{\partial^2 L}{\partial S^2}(S) \qquad \text{(or} \quad \Gamma_L(S,t) = \frac{\partial^2 L}{\partial S^2}(S,t), \quad \text{etc.)}$$

The greek Θ is the time derivative,

$$\Theta_L(S,t) = \frac{\partial L}{\partial t}(S,t).$$

There are other greeks. We mention here only vega, the first partial with respect to the volatility of S, by way of example, but we shall not study it: $\nu_L(S,\sigma) = \frac{\partial L}{\partial \sigma}(S,\sigma)$.

The Greeks are important in marginal analysis of portfolio values. When t is held fixed, and S changes from S to $S+\delta S$, the linear approximation to the change in the portfolio value is

$$\delta L(S,t) = L(S+\delta S,t) - L(S,t) \approx \frac{\partial L}{\partial S}(S,t)\delta S = \Delta_L(S,t)\,\delta S. \tag{1}$$

This will in general only be accurate for small price changes δS.

The second order approximation to the change in $L(S,t)$ is (recall Taylor's polynomial of order 2):

$$\delta L(S,t) \approx \frac{\partial L}{\partial S}(S,t)\delta S + \frac{1}{2}\frac{\partial^2 L}{\partial S^2}(s,t)\,(\delta S)^2 = \Delta_L(S,t)\,\delta S + \frac{1}{2}\Gamma_L(S,t)(\delta S)^2. \tag{2}$$

A second order approximation will be more accurate than a first order approximation and it will work well over a larger range of price changes δS.

If both time and price change, S by δS, t by δt, the linear approximation is

$$\begin{aligned}\delta L(S,t) &= L(S{+}\delta S, t{+}\delta t) - L(S,t) \approx \frac{\partial L}{\partial t}(S,t)\delta t + \frac{\partial L}{\partial S}(S,t)\delta S \\ &= \Theta_L(S,t)\delta t + \Delta_L(S,t) \end{aligned}\tag{3}$$

A second order approximation, suggested by Itô's rule and appropriate if we think that prices are modelled by Itô differentials, is

$$\delta L(S,t) \approx \Theta_L(S,t)\delta + \Delta_L(S,t)\delta S + \frac{1}{2}\Gamma_L(S,t)(\delta S)^2. \tag{4}$$

This is the second order approximation obtained using a Taylor polynomial of order two in δt and δL, but dropping the terms with $\delta t\,\delta L$ and $(\delta t)^2$.

Recall the delta, gamma and theta for the Black-Scholes price, derived in the derivation of the Black-Scholes formula. These are the greeks for the portfolio holding one option. The greeks for the call price are

$$\begin{aligned}\Delta_{C^{T,X}}(S,t) &= N(d_1(S,t)) &(5)\\ \Gamma_{C^{T,X}}(S,t) &= \frac{1}{\sigma S\sqrt{2\pi(T-t)}}e^{-d_1^2(t,s)/2} &(6)\\ \Theta_{C^{T,X}}(S,t) &= -Xe^{-r(T-t)}N(d_2(S,t)) - \frac{1}{2}\sigma^2 S^2\Gamma_{C^{T,X}}(S,t) &(7)\end{aligned}$$

The greeks for the Black-Scholes put price function are easily obtained from these using put-call parity.

According to (1), $\Delta_L(S,t)$ describes the first order sensitivity of the portfolio value to changes in the price S. A portfolio is **delta neutral** at S if $\Delta_L(S,t) = 0$. To first order, a fluctuation in the stock price does not change the value of a delta neutral portfolio; in such a portfolio, the risk from fluctuation of the price S will be relatively small. Hedgers and risk managers like to have delta neutral portfolios.

Example. Suppose Alice is short 10 calls. The value of this portfolio is $-10C(S)$, where C is the call price as a function of S—ignore t for the moment. Such a *naked* call position is exposed to the risk of fluctuation of S. If S increases from S to $S' = S + \delta S$, the price of the call increases from $C(S)$ to $C(S')$ and the value of Alice's portfolio decreases, the change in value being

$$-10(C(S') - C(S)).$$

As call prices increase as the current stock price increases, Alice will lose money if $S' > S$. As Alice has an obligation to honor the calls she has sold if they are exercised, she wants to protect against this risk. Can Alice create a delta-neutral portfolio by adding shares of XYZ Corp. to her portfolio, and if so, how many shares should she buy?

The answer is simple. Suppose that Alice buys a shares; the total value of her portfolio is then $L(S) = aS - 10C(S)$. Since $\Delta_L(S) = a - 10C'(S) = a - 10\Delta_C(s)$, her portfolio will be delta-neutral if

$$a = 10\Delta_C(S).$$

That is she should buy $\Delta_C(S)$ shares of stock per call option she owns.

Now, if you think about it, we know this result already. In deriving the Black-Scholes pde, we found that, denoting the price of the option by $v(S_t, t)$ where v satisfies the Black-Scholes pde, one constructs a replicating portfolio by holding $\dfrac{\partial v}{\partial s}(S_t, t)$ shares of stock at time t. In the derivation, this choice of the number of shares specifically served to remove the risk to the portfolio value due to fluctuation of the price.

The interesting point about the present analysis is that it applies without assumption on the specific nature of $C(S)$. We could do the same analysis on any derivative depending on the price of the stock, as in example (c). There the portfolio value was $L(S, t) = xS + zD(S, t)$, where D is the price of some derivative security. This portfolio is delta-neutral if

$$x = -z\Delta(S, t). \qquad \diamond$$

Numerical Example. Consider a stock trading at \$60 today, and a call option that expires in 3 months time with a strike of \$65. Assume that $r = 0.05$ and $\sigma = 0.3$.

(a) Alice is short 10 call contracts, each with 100 calls per contract, and holds nothing else. How much does she gain or lose if the stock price increases the next day to 61. How much does she gain or lose if the price drops to \$58 ?

The call price as a function of S and time is $C^{.25,65}(S,t)$; for notational convenience, write it as $C(S,t)$. Hence the value of the of the 10 call contracts to Alice is $L(S,t) = -1000C(S,t)$. The problem asks to compute $L(61, 1/365) - L(60,0) = 10(C(60,0)+C(61,1/365))$ and $L(58,1/365) - L(60,0) = 1000(C(60,0)+C(58,1/365))$. Using the Black-Scholes formula, the price is

$$
\begin{aligned}
C(S,0) &= SN\left(\frac{\ln(S/65)+(.095)(.25)}{.15}\right) - e^{-.0125}65N\left(\frac{\ln(S/65)+(.005)(.25)}{.15}\right)\\
C(60,0) &= 60N(-0.3753) - 64.1926N(-0.5253) = 1.985.
\end{aligned}
$$

(Actually I got this number using the Black-Scholes calculator in Maple 9.6.) Calculation also shows that $C(61, 1/365) = 2.338$. Hence

$$L(61,1/365) - L(60,0) = 1000(C(60,0)+C(61,1/365) = -\$353,$$

and Alice suffers a loss if the price rises a dollar.

On the other hand $C(58,1/365) = \$1.341$. In this case

$$L(58,1/365)-L(60,0) = 10(C(60,0)+C(58,1/365)) = \$1000\cdot(1.985-1.341) = \$644,$$

and Alice gains.

(b) Now suppose Alice creates a delta-neutral portfolio by buying $1000\Delta_C(60,0)$. Calculate the loss and gain on this portfolio value for the same scenarios as in (a).

Because $\Delta_C(60,0) = N(-0.3753) = 0.3537$, this portfolio will hold $1000{\cdot}N(0.3753) = 353.7$ shares of stock. Let us analyze the portfolio values and how they change in this circumstance. Today, since the stock trades at \$60,

$$L(60,0) = \$60(353.7) - 1000(1.985) = \$19237.00$$

If the price rises to \$61 the next day, the portfolio value is

$$L(61,1/365) = \$61(353.7) - 1000(2.338) = \$19,237.70.$$

Hence $\delta L = -\$0.70$, and Alice would lose only 70 cents. instead of the \$380 lost by the unhedged short calls.

On the other hand, if the price falls to \$58,

$$L(58,1/365) = \$58(353.7) - 1000(1.341) = \$19,173.60.$$

In this case, $\delta L = -\$63.40$, in contrast to the gain that would occur had there been no hedging.

(c) These numerical results and the delta hedging here deserve some discussion. We see that the delta-neutral portfolio is very much less sensitive to a change in the stock price than the unhedged portfolio. Still, there are small movements in the value of L and they are negative whether the stock price increases or decreases. This can be explained theoretically. First of all notice that the portfolio price movement we are tracking in (b) incorporates changes to portfolio value from changes in and time—going from today to tomorrow—as well as price. So, in this example we are studying a change in value of the form

$$\delta L = L(S{+}\delta S, t{+}\delta t) - L(S, t).$$

with $t{=}0$, $\delta t{=}1/365$, $S{=}60$ and $\delta S{=}1$ or $\delta S{=}-2$. The most accurate approximation to this difference among those listed in equations (2), (3), and (4) is the last one (4), which is

$$\delta L(S, t) \approx \Theta_L(S, t)\delta t + \Delta_L(S, t)\delta S + \frac{1}{2}\Gamma_L(S, t)(\delta S)^2.$$

For our particular example, $\Delta_L(60, 0) = 0$ and so the second order approximation becomes to

$$\delta L = L(60{+}\delta S, t{+}1/365) - L(60, 0) \approx \Theta_L(60, 0)\frac{1}{365} + \frac{1}{2}\Gamma_L(60, 0)(\delta S)^2.$$

Thus, while the delta-neutral hedge does take care of the first order contribution to δL of δS, it ignores the effects of time change and of contributions of second order in δS. This is why the delta-neutral hedge is not perfect.

It is interesting to study the second order approximation numerically. Since

$$L(S, t) = S \cdot 1000 \cdot \Delta_C(60, 0) - 1000C(S, t),$$

it is easy to see that $\Theta_L(60, 0) = -1000\Theta_C(60, 0)$ and $\Gamma_L(60, 0) = -1000\Gamma_C(60, 0)$. Using the formulae (6) and (7), one finds $\Gamma_L(60, 0) = -1000(0.0413) = 41.3$ and $\Theta_L(60, 0){=}-1000(-7.653){=}21$, so as a function of δS with δt fixed at 1,

$$\delta L \approx \frac{7653}{365} - (41.3/2)(\delta S)^2 = 20.97 - (41.3/2)(\delta S)^2. \tag{8}$$

In the case that the price drops to \$58, $\delta S = -2$ and then the approximation is $\delta L \approx 20.97 - 2(41.3) = -61.63$, compared to an actual δL of -63.40. If the price moves up to \$61, so that $\delta S = 1$, $\delta L \approx 20.97 - 41.3/2 = -0.32$, compared to an actual value of -0.70. The approximation (8) thus captures the real changes in δL pretty accurately.

Looking at (8) one sees that, because the coeffiecient of the quadratic term is negative, δL will be negative (approximately) unless $|dS| < \sqrt{2(21)/(41.3 \cdot 365)} = .053$, that is, unless δS is quite small. For reasonably sized price changes, the portfolio value will decline slightly whether S moves up or down. In the final analysis, this is due to the convexity of the price curve $C(S, 0)$ as a function of S. To see this, observe from (6) that $\Gamma_C(S, t)$ is always positive; this implies convexity in S in C and, because $\Gamma_L(S, t) = -1000\Gamma_C(s, t)$, concavity of $L(S, t)$ in S. This ends the numerical example. $\diamond$

We return to the general discussion. As we have said, delta neutral hedging provides protection to first order in the change of portfolio value due to price change. More accurate hedging schemes are suggested by the second order approximation:

$$\begin{aligned} \delta L(S) = L(S+\delta S) - L(S) &\approx \frac{\partial L}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 L}{\partial S^2}(\delta S)^2 \\ &= \Delta_L(S)\delta S + \frac{1}{2}\Gamma_L(S)(\delta S)^2. \end{aligned} \tag{9}$$

(For this discussion we are ignoring dependence on time.) If L is delta neutral, $\delta_L \approx (1/2)\Gamma_L(S, t)(\delta S)^2$. When δS is small this quantity is relatively small, but when $(\delta S)^2$ is large, it can be significant, and if $\Gamma_L(S, t) < 0$, will be negative and hence mean a significant loss of portfolio value. To return for a moment to the numerical example, if the price were to increase in one day to \$65, the delta neutral portfolio would change by $L(65, 1/365) - L(60, 0) = 65 \cdot 353.7 - 1000(4.274) - 19242 = -525.50$. One could eliminate this risk also if one could also hedge Γ_L. Following this idea, it is natural to introduce the idea of a *delta-gamma neutral portfolio*. This is a portfolio for which both $\delta_L(s, t)$ and $\Gamma_L(S, t) = 0$. Such portfolio will not change its value if the price S shifts, at least to the order of accuracy of the approximation (9).

Example: Creating a delta-gamma neutral portfolio to hedge a short call position.. Assume a position short 1 calls at strike X with expiration T. We want to add securities to this portfolio to make it delta-gamma neutral. It is not possible to do this simply by buying shares of stock, as this adds a only term of the form xS, which is linear; the second derivative with respect to S of x is zero, and so cannot contribute a term to cancel the gamma of the call option and so render the combined portfolio delta-gamma neutral. Instead one needs to add another derivative security based on the stock price S. One method is to purchase another option written on the same stock, but being of a different type or having a different expiration and strike. In this example, we explore hedging against a short position in calls using stocks and additional call options with different strike and expiration.

Let the original call option have strike X and expiration T. The portfolio will be short one of these call options. To it, we add x shares of stock, and, in addition, z calls at a different strike X' and expiration T'. The value of this portfolio is

$$L(S,t) = xS - C^{T,X}(S,t) + zC^{T',X'}(S,t).$$

The object is to choose x and z so that the portfolio is delta-gamma neutral. This is easily done. By taking partial derivatives,

$$\begin{aligned} \Delta_L(S,t) &= x - \Delta_{C^{T,X}}(S,t) + z\Delta_{C^{T',X'}}(S,t) \\ \Gamma_L(S,t) &= -\Gamma_{C^{T,X}}(S,t) + z\Gamma_{C^{T',X'}}(S,t) \end{aligned}$$

Therefore to get $\Gamma_L(S,t) = 0$ one needs to choose z so that

$$z = \frac{\Gamma_{C^{T,X}}(S,t)}{\Gamma_{C^{T',X'}}(S,t)}$$

Once z is so chosen, $\Delta_L(S,t)$ will be zero if

$$x = \Delta_{C^{T,X}}(S,t) - \Delta_{C^{T',X'}}(S,t)\frac{\Gamma_{C^{T,X}}(S,t)}{\Gamma_{C^{T',X'}}(S,t)}.$$

For such a strategy to be practicable and in order that the purchase of the z new calls is relatively inexpensive, one would like to find T' and X' so that $\Gamma_{C^{T',X'}}(S,t)$ is large relative to $\Gamma_{C^{T',X'}}(S,t)$, so that z is small. If we assume Black-Scholes pricing, this can be achieved with a call that expires shortly at a strike close to the current stock price. Indeed, if $S = X'$, then from equation (6) and the definition of $d_1(S,t)$, one easily derives

$$\Gamma_{C^{T',X'}}(X',0) = \frac{1}{X'\sigma\sqrt{2\pi}} \exp\{-T'(r+\sigma^2/2)^2/\sigma^2\}.$$

This will be fairly large if T' is small, since the denominator is then small and the argument of the exponent is then close to zero.

The following continuation of the numerical example above illustrates a delta-gamma neutral hedge of this sort.

Example, continued. In the numerical example above Alice was hedging a postion of 1000 calls of price $C^{.25,65}(60,0) = 1.98$; recall that its gamma and delta are

$$\Delta_{C^{.25,65}} = 0.3537, \qquad \Gamma_{C^{.25,65}}(60,0) = 0.0413.$$

Suppose calls on the same stock are available that expire in 15 days at strike 60; its current price is $C^{1/24,60}(60,0) = 1.50$ and its gamma and delta are

$$\Delta_{C^{1/24,60}} = 0.5239, \qquad \Gamma_{C^{1/24,60}}(60,0) = 0.1091.$$

A delta-gamma neutral portfolio that is short 1000 calls at strike 65 in 3 months can be constructed by going long

$$z = 1000\frac{0.0431}{0.1091} = 395$$

of the calls that expire in 15 days at a strike of 60 and purchasing

$$x = 1000\left(0.3537 - (0.5239)\frac{0.0431}{0.1091}\right) = 146.7$$

shares of stock. Buying the short expiration calls is not expensive; they cost $\$395 \times 1.50 = \$592.50$.