640:495 Mathematical Finance: More about Black-Scholes, pde analysis and Greeks.

I. A further comment on the Greeks and Black-Scholes pricing.

Consider a derivative security on an underlying asset that pays $U(S_T)$ at time T, S_T being the asset price at time T. If the price process $\{S_t, 0 \leq t \leq T\}$ is assumed to follow a Black-Scholes model with volatility σ, the no-arbitrage price of the derivative at time t is given by $V(t) = u(S_t, t)$ where u is a solution to the Black-Scholes partial differential equation:

$$\begin{aligned} \frac{\partial u}{\partial t}(S,t) + rS\frac{\partial u}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2}(S,t) - ru(S,t) &= 0; \quad S > 0,\ 0 \leq t < T; \quad (1) \\ u(S,T) &= U(S). \quad (2) \end{aligned}$$

The Black-Scholes pde (1) can be rewritten in terms of the greeks of the price $u(S,t)$ and so implies a standing relationship between their values:

$$\Theta_u(S,t) + rS\Delta_u(S,t) + \frac{1}{2}\sigma^2 S^2 \Gamma_u(S,t) - ru(S,t) = 0. \quad (3)$$

So, when working in the Black-Scholes framework, one can deduce the value of one of these greeks if the other two are known.

To illustrate for yourself go to an online Black-Scholes price calculator and check. For example at `http://www.option-price.com/index.php`, I entered $X = 48$, $S = 50$, $t = 90$ days, $r = 5\%$ and $\sigma = 30\%$ ($\sigma = 0.3$) and asked for rounding to 4 decimal places. The calculator gives a call price of 4.3925, a delta of 0.6664, and a Theta of $-0.019(365) = 6.935$. (Note: this option calculator give daily theta, and the number must be multiplied by 365 because the theta in the Black-Scholes pde must be denominated in the unit of time which is used for the volatility and interest rate, and this is yearly. Plugging these into equation (3),

$$-0.0189 + (.05)\cdot(50)\cdot(.6664) + (.045)\cdot(50)^2\Gamma - (0.05)(2.2697) = 0.$$

Solving for Γ, gives $\Gamma = 0.0488$, which matches the gamma given by the option calculator.

II. A heuristic for Itô's rule applied to price processes.

Let $\{S_t\}$ denote the price of an asset. Let $g(S,t)$ be a function of price and time. Itô's rule implies the following heuristic for how $g(S_t,t)$ changes over a small increment of time:

$$dg(S_t,t) = \frac{\partial g}{\partial t}(S_t,t)\,dt + \frac{\partial g}{\partial S}(S_t,t)\,dS_t + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}\,(dS_t)^2. \quad (4)$$

The student should compare this to the second order approximation formula discussed in the previous set of notes on Portfolios and Greeks—see the approximation in equation (4) of those notes. (This approximation was illustrated in the last example of these notes.)

Typically, price models in continuous time are given in the form $dS_t = \alpha_t\, dt + \beta_t\, dB_t$ where B is a Brownian motion. In this case, one may obtain the Itô rule proper by replacing $(dS_t)^2$ in (4) by $\beta_t^2\, dt$, because the expansion of $(dS_t)^2$ is $\alpha_t^2\, dt + 2\alpha_t\beta_t\, dt\, dS_t + \beta_t^2(dB_t)^2$, and Itô tells us to throw away the terms with $dt\, dS_t$ or $(dt)^2$ and replace $(dB_t)^2$ by dt. Thus, from (4) we recover:

$$
\begin{aligned}
dg(S_t, t) &= \frac{\partial g}{\partial t}(S_t, t)\, dt + \frac{\partial g}{\partial S}(S_t, t)\ [\alpha_t\, dt + \beta_t\, dB_t] + \frac{1}{2}\frac{\partial^2 g}{\partial S^2}(S_t, t)\beta_t^2\, dt \\
&= \left\{\frac{\partial g}{\partial t}(S_t, t) + \alpha_t\frac{\partial g}{\partial S}(S_t, t) + \frac{1}{2}\beta_t^2\frac{\partial^2 g}{\partial S^2}(S_t, t)\right\} dt + \beta_t\frac{\partial g}{\partial S}(S_t, t)\, dB_t \quad (5)
\end{aligned}
$$

III. Generalizing the Black-Scholes model.

The Black-Scholes price model with drift μ and volatility σ is $S_t = S_o \exp\{(\mu - \sigma^2/2)t + \sigma B_t\}$. We have seen by applying Itô's rule that this model is equivalent to

$$dS_t = \mu S_t\, dt + \sigma S_t\, dB_t.$$

To understand some of the underlying assumptions behind this model, divide both sides of the equation by S_t:

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dB_t. \tag{6}$$

The quantity on the left hand side, the ratio of the change of price over a succeeding small time increment with the current price, is an incremental rate of return. The left hand side is the sum of a small deterministic movement of size $\mu\, dt$ and a random fluctuation which one may think of as a normal random variable with mean 0 and standard deviation $\sigma\sqrt{dt}$. Equation (6) says that this incremental rate of return is independent of the current price level. For example, suppose the daily volatility is %2 and daily drift is \$0.03 for a certain stock. The relative return will fall within two standard deviations of its mean with probability about .95; the interval two standard deviations about the mean is $[0.03+0.04, 0.03+0.04] = [-0.01, 0.07]$. This interval gives the order of the day to day fluctuations of the incremental return. If the stock is trading around \$5 the numerical values of the price fluctuation itself are 5 times these returns, that is fluctations lying in the interval from $\$-.05$ to \$.35. However, if the stock is trading around \$10, the fluctuations of the price itself will be twice as great, ranging with probability 0.95 in $[-.10, 0.70]$. Thus for the Black-Scholes model, the price fluctuations scale with the price; for a price two times as large, the fluctuations are two times as large; if the price is three times as large, the fluctuations are three times as large, etc.

One can imagine that more accurate price models could be built by allowing the drift and volatility of a stock to depend on the price level. Certainly this would be more general, and probably more realistic. Thus, one would replace μ by a function $m(S)$, and σ by a function $\gamma(S)$ and write a price model in the form:

$$dS_t = m(S_t)\,dt + \sigma\gamma(S_t)S_t\,dB_t. \tag{7}$$

The first question one most ask is whether this makes mathematical sense. In other words, given m and γ can we find some process S_t that depends at each t on the Brownian motion path up to time t and that satisfies this equation. We did not have to answer this question for the Black-Scholes model because we had an explicit formula for S_t in terms of the Brownian motion B. For most equations of the form (7) we cannot find such an explicit formula, but a theory for (7), known as the theory of stochastic differential equations, says that, yes, (7) admits solutions under only mild conditions on m and σ. We will not worry about this. Our main point is to bring up the generalization because it is important in applications in which quants try to build better models.

But there is a very important point we wish to make. In general, if (7) is our reference model for an asset price, one is not able to derive option prices in an explicit form, like the Black-Scholes formula. However, the technique for deriving the Black-Scholes pde generalizes immediately and easily to these more general models, allowing one derive a partial differential equation for the price function. Since these pde's are amenable to numerical solution by fairly standard techniques, we have a practical tool to get option prices with (hopefully) more accurate models having price dependent volatilities.

As an example, we will solve homework problem 89. Consider a derivative security that pays (S_T) at time T. The steps of the method, as we have applied them previously are

(i) Replace $m(S)$ in (7) by rS. This is the risk-neutrality assumption.

(ii) Assume that we can find a replicating portfolio Π_t for the derivative such that Π_t has the form $\Pi_t = u(S_t, t)$. This requires that

$$u(S,T) = U(S) \tag{8}$$

and

$$du(S_t,t) = d\Pi_t = (\Pi_t - \phi_t S_t)r\,dt + \phi_t\,dS_t, \tag{9}$$

for some hedging function ϕ_t that represents the number of shares of stock held at time t.

(iii) apply Itô's rule to evaluate the left hand side of (9) and find a partial differential equation for $u(S,t)$ and an expression for the hedging function ϕ_t by equation coefficients of dt and dB_t in (9).

Let us carry this out for the price process

$$dS_t = rS\,dt + \sigma\sqrt{S_t}\,dS_t.$$

First observe that, using this in (9) and using the assumption $\Pi_t = u(S_t,t)$,

$$d\Pi_t = (\Pi_t - \phi_t S_t)r\,dt + \phi_t\,dS_t = ru(S_t,t)\,dt + \phi_t\sigma\sqrt{S_t}\,dB_t, \tag{10}$$

since the $\phi_t r S_t$ term from $\phi_t\,dS_t$ cancels out the same term from $(\Pi_t - \phi_t S_t)r\,dt$.

Now apply Itô's rule as in (5) but with $\alpha_t = rS_t$ and $\beta_t = \sigma\sqrt{S_t}$

$$du(S_t,t) = \left\{\frac{\partial u}{\partial t}(S_t,t) + rS_t\frac{\partial u}{\partial S}(S_t,t) + \frac{1}{2}\sigma^2 S_t\frac{\partial^2 u}{\partial S^2}(S_t,t)\right\}dt + \sigma\sqrt{S_t}\frac{\partial u}{\partial S}(S_t,t)\,dB_t.$$

By equating the coeffiecient's of dB_t in this expression and in (10), we get $\phi_t\sigma\sqrt{S_t} = \sigma\sqrt{S_t}\frac{\partial u}{\partial S}(S_t,t)$, and for this to be true we require

$$\phi_t = \frac{\partial u}{\partial S}(S_t,t). \tag{11}$$

(This much we can automatically expect by the delta hedging principle!) Now equate the dt coefficients:

$$\frac{\partial u}{\partial t}(S_t,t) + rS_t\frac{\partial u}{\partial S}(S_t,t) + \frac{1}{2}\sigma^2 S_t\frac{\partial^2 u}{\partial S^2}(S_t,t) = ru(S_t,t).$$

This requires that u satisfy the partial differential equation

$$\frac{\partial u}{\partial t}(S,t) + rS\frac{\partial u}{\partial S}(S,t) + \frac{1}{2}\sigma^2 S\frac{\partial^2 u}{\partial S^2}(S,t) - ru(S,t) = 0, \quad S>0,\ 0\le t<T. \tag{12}$$

Of course u must also satisfy (8); equations (8) and (12) together constitute the pde with terminal condition characterizing the price. (In addition, one can also demand $u(0,t)\equiv 0$, since if the price hits zero the company goes bankrupt and the option cannot be exercised.) $\diamond$

There was nothing too special about $\sqrt{S_t}$ in this exercise. It is easy to follow through and see that if $dS_t = rS_t\,dt + \gamma(S_t)S_t\,dS_t$, the pde is simply Black-Scholes with σ replaced by $\gamma(S)$;

$$\frac{\partial u}{\partial t}(S,t) + rS_t\frac{\partial u}{\partial S}(S,t) + \frac{1}{2}\gamma^2(S)S^2\frac{\partial^2 u}{\partial S^2}(S,t) - ru(S,t) = 0, \quad S>0,\ 0\le t<T. \tag{13}$$