610)621: Mathematical Finance: Notes for Lectures 8 and 9.

I. Orientation.

So far in this course we have done the following:

1. For the *one-period, binomial model* we used the principles of no-arbitrage and portfolio replication to derive a formula that gives the price of an arbitrary derivative.

2. We expressed the one-period pricing formula as a discounted expectation of the derivative pay-off with respect to the *risk-neutral measure.* The risk-neutral measure is an assignment of probabilities to the price movements so that $S_0 = e^{-r\tau}\tilde{E}\left[S_\tau\right]$; here τ is the duration of the period, r the nominal interest rate, S is the price of the stock, and $\tilde{E}$ denotes expectation using the risk-neutral probabilities.

3. For the multi-period, binomial tree model, we stated a backward induction (dynamic programming) algorithm to compute the price of any derivative expiring after period N. The algorithm gives the price of the derivative at any period k before N, given the market history up to period k.

Our next project is to do for the multi-period model what we did for the one-period model—justify the backward induction pricing algorithm using the ideas of no-arbitrage and portfolio replication, and, if possible, express derivative prices as expectations.

II. The binomial tree model with N periods.

In words, the assumptions of he N-period, binomial tree model are

1. In each period the market either goes up (u) or down (d).

2. There is one risky asset, called the stock. If the market goes up in a period the return over that period on the asset price is g, if it goes down the return is ℓ, and ℓ and g are the same for each period.

3. A dollar invested over one period at the risk free rate grows to $e^{r\tau}$ dollars; τ is the duration of each period (in years) and r is the nominal, annual risk-free rate.

4. It is assumed that $\ell < e^{r\tau} < g$. As we know, this rules out arbitrage trading between the stock and investing at the risk-free rate.

In lecture 1 (see class notes) we already began to discuss notation for this model in the cases $N = 1$ and $N = 2$. Homework exercise #4 extended the notation to $N = 3$. Here, general notation and an important formula for the price process will be established.

A market history (or outcome, or path) for the N-period model is a sequence $\omega = (w_1, w_2, \ldots, w_N)$ of length N, of u's and d's, $w_i = u$ signifying an upswing in period i, $W_i = d$ a downswing. The set of all market histories of length N is denoted Ω, or, when we want to be clear about the number of periods, Ω_N. Thus, as in the first lecture, $\Omega_2 = \{(u, u), (u, d), (d, u), (d, d)\}$.

Prices and payoff functions of derivatives are all functions on the set of market histories Ω. The price of the stock at the beginning of the period is denoted S_0 and is assumed given. For each integer t, $1 \le t \le N$, and market outcome ω, $S_t(\omega)$ denotes the price of the stock at the end of period t given market history ω. Note that the assumption 2 above, implies that the price S_t depends only on the market history in the first t periods. Therefore, we shall write $S_t(\omega)$ as

$$S_t(w_1, \ldots, w_t),$$

keeping only those variables on which S_t truly depends. Applying this principle and assumption 2, one calculates

$$\begin{array}{cccc} S_1(u) = gS_0, & S_1(d) = \ell S_0, & S_2(u, u) = g^2 S_0, & S_2(u, d) = S_2(d, u) = g\ell S_0, \\ S_2(d, d) = \ell^2 S_0, & S_3(u, u, u) = g^3 S_0, & \multicolumn{2}{c}{S_3(u, u, d) = S_3(u, d, u) = S_3(d, u, u) = g^2 \ell S_0,} \end{array}$$

and so on. Note that this notation differs from what was used in the lecture 1 notes and in the solution to homework exercise #4: for example in the 3-period model, we previously wrote $S_1(u, u, u)$, but now we drop the superfluous variables indicating market movements in periods beyond period 1 and write only $S_1(u)$.

To generalize, if a function V on Ω depends only on the first t market movements of the full history $\omega = (w_1, \ldots, w_N)$, we shall write $V(w_1, \ldots, w_t)$.

Although we have not yet put probabilities on Ω, we think of the market outcomes as random. With this is mind, it is useful to think of functions on Ω as random variables. Here, we make an important convention that we try to stick to for clarity. When we want to refer to the value of the function V for a specific market outcome $\omega = (w_1, \ldots, w_N)$ we write, of course, $V(w_1, \ldots, w_N)$. When we want to think about V as a function on Ω, that is, as a random variable whose value will only be determined, once a market history is observed, we write just V.

The sequence of prices $S_0, S_1, S_2, \ldots, S_N$, understood in the random variable sense, will be refered to as the *price process.*

Now that you have, I hope, digested the notation, we discuss a useful formula for the price process. For each t, define the function

$$Y_t(w_1, \dots, w_t) = \text{the number of upswings } (u\text{'s}) \text{ in } (w_1, \dots, w_t).$$

Of course, since $(w_1, \dots, w_t)$ is a sequence of length t, $t - Y_t(w_1, \dots, w_t)$ is the number of downswings (d's) in $(w_1, \dots, w_t)$. Thus, for example $Y_5(u, u, d, u, d) = 3$, $Y_5(d, d, d, u, d) = 1$.

Consider now $S_t(w_1, \dots, w_n)$; every upswing changes the price of the asset by a factor of g, every downswing by a factor of ℓ. At time zero the price is S_0; thus

$$S_t(w_1, \dots, w_t) = g^{Y_t(w_1,\dots,w_t)} \ell^{t - Y_t(w_1,\dots,w_n)} S_0, \qquad \text{for all } (w_1, \dots, w_t). \tag{1}$$

Summarized more elegantly, this is an identity of funcitions:

$$S_t = g^{Y_t} \ell^{t - Y_t} S_0.$$

An easy consequence of this identity is that the possible values of S_t are $\{g^t S_0, g^{t-1}\ell S_0, g^{t-2}\ell^2 S_0, \dots, g\ell^{t-1}S_0, \ell^t S_0\}$.

III. Probabilities and expectations on the market model.

This section is in preparation for defining a risk-neutral measure on the multi-period, binomial tree model. Again, simple examples of what we shall do in this section have already been given in the lecture 1 notes and in howework problem #4 for the cases $N = 2$ and $N = 3$.

One could assign probabilities to the market outcomes in as complicated a manner as one wishes. However, we will consider only a special class of probability assignments. Fix a number p, $0 < p < 1$. We will study only probability assignments determined by the following rule.

The random walk probability model: In each period, the probability of an upswing (u) is p and of a downswing (d) is $q = 1 - p$. Market movements in different periods are independent.

Let us use the notation $\mathbb{P}_p$ to denote probabilities calculated according to this rule when p is the probability of upswing. For example,

$\mathbb{P}_p\left((w_1, \dots, w_N)\right)$ denotes the probability that the market history is $(w_1, \dots, w_N)$.

These probabilities were calculated in problem #4 for the three period model. Using the assumption of independence of different periods,

$$\mathbb{P}_p\left((u, u, u)\right) = p^3, \quad \mathbb{P}_p\left((u, u, d)\right) = p^2 q, \quad \mathbb{P}_p\left((d, d, u)\right) = pq^2, \quad \text{and so on.}$$

Because each movement up contributes a factor of p and each movement down a factor of q, the general formula is

$$\mathbb{P}_p\left((w_1,\ldots,W_N)\right) = p^{Y_N(w_1,\ldots,W_N)} q^{N-Y_N(w_1,\ldots,W_N)}, \tag{2}$$

using the function Y_t defined above which counts the number of u's in a sequence.

A cultural aside: Random walk is a fundamental model of probability theory applied in a wide variety of circumstances, and you likely studied it in your probability course. Here, think of the movements up or down as corresponding to the steps of a drunkard as he staggers out of a bar. Steps in the "up" direction take him closer to home, steps in the "down" direction toward the cop in the opposite direction on the street corner. The probability p is a measure of his inebriation. He only manages to take a step in the right direction with probability p; having taken a step, he is unaware if it is toward or away from home and it does not affect the direction of his next step. As a model for the movements of a stock market, random walk goes back to a thesis of Bachelier, written in the early twentieth century.

To continue the discussion of $\mathbb{P}_p$, if A is a subset of possible market outcomes, $\mathbb{P}_p(A)$ denotes its probability. To get $\mathbb{P}_p(A)$ one sums $\mathbb{P}_p(\omega)$ over all outcomes in A. We'll see examples in a moment. We refer to $\mathbb{P}_p$ as a probability measure on Ω, the set of possible outcomes; "probability measure" is math lingo for a rule assigning probabilities to events.

Given a probability assignment $\mathbb{P}_p$ on Ω, functions on Ω truly become random variables. There are two important facts to state right away.

(A) If the probability measure $\mathbb{P}_p$ is used, then, for each t, Y_t is a binomial random variable with parameter t and p; this means explicity that the probability mass function of Y_t is

$$\mathbb{P}_p\left(Y_t = k\right) = \binom{t}{k} p^k q^{t-k} \tag{3}$$

This is easy to see. Think of the market movements as coin flips, with u corresponding to heads and d to tails. Using $\mathbb{P}_p$, Y_t is the total number of heads in t *independent* tosses, with p being the probability of heads. There are $\binom{t}{k}$ sequences of length t with exactly k heads and $t-k$ tails and each such sequence has probability $p^k q^{t-k}$, so the probability of k heads in t tosses is $\binom{t}{k} p^k q^{t-k}$, as in (3).

(B) For any t, the possible values of S_t are $g^k \ell^{t-k} S_0$, $0 \le k \le t$, and

$$\mathbb{P}_p\left(S_t = g^k \ell^{t-k} S_0\right) = \binom{t}{k} p^k q^{t-k}, \quad 0 \le k \le t. \tag{4}$$

This is an immediate consequence of the fact that Y_t is binomial and of the representation of $S_t = g_y^Y \ell^{t-Y_t} S_0$ found in (1).

Fact B allows us to calculate the expected values of prices. Expectation, calculated using $I\!P_p$ will be denoted $E_p[\cdot]$. By the definition of expected value, if X is any function of the random market path and if the possible values of X are $x_1, x_2, \ldots, x_K$, then

$$E_p[X] = x_1 I\!P_p(X=x_1) + x_2 I\!P_p(X=x_2) + \cdots + x_K I\!P_p(X=x_K). \tag{5}$$

(A sloppy, but very convenient, way to write this definition is

$$E[X] = \sum_x x I\!P_p(X=x).$$

We will use this form often.)

Expected values of prices. The simplest calculation is the expected value of $E_p[S_1]$:

$$E_p[S_1] = \ell S_0 I\!P_p(S_1=\ell S_0) + g S_0 I\!P_p(S_1=gS_0) = \ell S_0 I\!P_p(w_1=u) + g S_0 I\!P_p(w_1=d) = S_0(\ell q + gp). \tag{6}$$

The *expected return* on the stock in one period is thus

$$\frac{E[S_1]}{S_0} = \ell q + gp = \ell(1-p) + gp.$$

What about $E_p[S_t]$ for general t? We use the information derived above in fact B. Since the possible values of S_t are $g^k \ell^{t-k} S_0$, $0 \le k \le t$, and since the probability that S_t equals $g^k \ell^{t-k} S_0$ is $\binom{t}{k} p^k q^{t-k}$,

$$\begin{aligned} E_p[S_t] &= \sum_{k=0}^{t} S_0 g^k \ell^{t-k} S_0 \binom{t}{k} p^k q^{t-k} \\ &= S_0 \sum_{k=0}^{t} \binom{t}{k} (pg)^k (\ell q)^{t-k} \\ &= S_0(\ell q + gp)^t. \end{aligned} \tag{7}$$

The last step is an application of the binomial theorem: $(x+y)^n = \sum_0^n \binom{n}{k} x^k y^{n-k}$. Notice what equation (7) says; we just interpreted the factor $(\ell q + gp)$ as the expected return over one period. Equation (7) says that the expected return over t periods is the expected return per period, raised to the power t: $(\ell q + gp)^t$. Since market movements in different periods are independent, this is exactly what we should expect.

The next example is a worked problem to get more experience with computing expectations in the random walk model.

Example. Consider a 4-period model with $S_0 = 32$, $g = 5/4$, $\ell = 3/4$, and $p = 1/3$. Let $\max\{S_4 - 35, 0\}$ be the payoff a European call option at strike 35 that expires at the end of the fourth period. Find $E_p[\max\{S_4 - 35, 0\}]$. (If you believed that upswings had probability 1/3, this would be the payoff you would expect on average from the option.)

The three highest possible prices of S_4 are

$$32(5/4)^4 = 78.125, \quad 32(5/4)^3(3/4) = 48.875, \quad 32(5/4)^2(3/4)^2 = 28.125.$$

The last price is below the strike, as are all the other prices, and the option will expire worthless if they occur. Thus,

$$\begin{aligned} E_p[\max\{S_4 - 35, 0\}] &= (78.125 - 35)\mathbb{P}_p(S_4 = 78.125) + (48.875 - 35)\mathbb{P}_p(S_4 = 48.875) \\ &\qquad + 0 \cdot \mathbb{P}_p(S_4 \leq 35) \\ &= 43.125(1/3)^4 + (13.875)\binom{4}{3}(1/3)^3(2/3) = 1.90. \end{aligned}$$

Before leaving this topic, it is useful to mention another formula that gives the expectation. The formula (5) only relied on the underlyling market history probabilities to compute $\mathbb{P}_p(X = x)$. This next formula works directly with X as a function on the market histories: let X be a function on the market history space Ω_N; then

$$E_p[X] = \sum_{\omega} X(\omega)\mathbb{P}_p(\omega). \tag{8}$$

In this formula, the sum is to be interpreted as over all market histories ω in Ω.

This formula makes direct intuitive sense if you think about it; it says the expected value is thesum of the values $V(\omega)$ weighted by the probability that ω is the market history. To understand how formula (8) works and why it gives the right result, consider first the example that we just did. Let $V = \max\{S_4 - 35, 0\}$. Write out V as an explicit function of the market outcomes:

$$\begin{aligned} &V(u,u,u,u) = 78.125 - 35 = 43.125 \\ &V(u,u,u,d) = V(u,u,d,u) = V(u,d,u,u) = V(d,u,u,u) = 48.875 - 35 = 13.875 \\ &V(w_1,w_2,w_3,w_4) = 0 \quad \text{for all other } (w_1,w_2,w_3,w_4). \end{aligned}$$

This is because, for instance, $V(\omega) = 43.125$ precisely when $S_4(\omega) = 78.125$, which occurs only if $\omega = (u,u,u,u)$, while $V(\omega) = 13.875$ precisely when $S_4(\omega) = 48.875$, which occurs only when ω belongs to the set $\{(u,u,u,d), (u,u,d,u), (u,d,u,u), (d,u,u,u)\}$.

Start to write out the sum represented by the right side of (8) when X is replaced by V. The full sum has 16 terms, one for each of the possible market histories of the four period model. We'll write out at least the non-zero terms:

$$\begin{aligned}
V(u,u,u,u)\mathbb{P}_p((u,u,u,u)) + V(u,u,u,d)\mathbb{P}_p((u,u,u,d)) + V(u,u,d,u)\mathbb{P}_p((u,u,d,u)) \\
+V(u,d,u,u)\mathbb{P}_p((u,d,u,u)) + V(d,u,u,u)\mathbb{P}_p((d,u,u,u)) + \cdots \\
= 43.125(1/3)^4 + 13.875(1/3)^2(2/3) + 13.875(1/3)^2(2/3) \\
+13.875(1/3)^2(2/3) + 13.875(1/3)^2(2/3).
\end{aligned}$$

Comparing to the example, it is clear that the answer is the same, and the reason is that the sum over the last four terms is precisely 13.875 times the probability that V equals 13.875.

Building on the insight of this example, let's show why in general (5) and (8) both compute the same thing. Imagine then that X is a function on the space Ω. For convenience, suppose it takes 3 values, x_1, x_2, and x_3. Let A_1 be the set of all ω such that $X(\omega)=x_1$, let A_2 be the set of all ωsuch that $X(\omega) = x_2$, and let A_3 be the set of all ω such that $X(\omega) = x_3$. Since X has only these values A_1, A_2 and A_3 form a partition of ω. Now start with the sum in (8) and partition the sum into which A_i ω falls:

$$\sum_{\omega} X(\omega)\mathbb{P}_p(\omega) = \sum_{\omega\in A_1} X(\omega)\mathbb{P}_p(\omega) + \sum_{\omega\in A_2} X(\omega)\mathbb{P}_p(\omega) + \sum_{\omega\in A_2} X(\omega)\mathbb{P}_p(\omega).$$

But the probability $p_p(A)$ of a set A of market histories is precisely $\mathbb{P}_p(A) = \sum_{\omega\in A} \mathbb{P}_p(\omega)$; and, by definition, $X(\omega) = x_i$ if $\omega \in A_i$. Thus, the last expression equals

$$x_1\mathbb{P}_p(X=x_1) + x_2\mathbb{P}_p(X=x_2) + x_3\mathbb{P}_p(X=x_3),$$

and this is precisely $E_p[X]$. The reader will notice that there is nothing special about the measure $\mathbb{P}_p$ that was used in this argument; the equivalence between (5) and (8) holds if p_p is replaced by any assignment of probabilities to the outcomes ω whatsoever.

It is often useful, both in theory and computation, to use (8) instead of (5) and that is why we have spent some time on it.

An aside for the probability theory nerd. In the advanced theory of probability, (8), or rather, an abstruse generalization of it, is taken as *the* basic definition of expectation, and (5) is deduced as a consequence, rather than the other way round, as in elementary probability.

IV. The risk-neutral measure on the binomial tree model.

A Risk-neutral measure for the one-period model recalled.

Recall the definition of the risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$ in the one period model: $\tilde{p}$ and $\tilde{q} = 1 - p$ are uniquely defined by the condition,

$$S_0 = e^{-r\tau} \left[\ell \tilde{q} + g \tilde{p} \right] = e^{-r\tau} \left[\ell (1 - \tilde{p}) + g \tilde{p} \right]. \tag{9}$$

The no arbitrage condition $\ell < e^{r\tau} < g$, guarantees that a unique $\tilde{p}$ exists and that $\tilde{p}$ and $\tilde{q}$ are both positive and As we pointed out in previous lectures, the right-hand side of (9) defines the expectation of S_1 if $\tilde{p}$ is set to the probability of u and $\tilde{q}$ to the probability of d; we wrote (we had S_τ but now S_1 is used in place of S_τ)

$$S_0 = e^{-r\tau} \tilde{E}[S_1] \tag{10}$$

Multiplying on both side by $e^{r\tau}$ and dividing by S_0:

$$\ell \tilde{q} + g \tilde{p} = \frac{\tilde{E}[S_1]}{S_0} = e^{r\tau}. \tag{11}$$

This identity says that the *expected return on the stock under the risk-neutral probabilities is the same as for the risk-free interest rate.* This is fundamental. You should make it your mantra! It explains the terminology of "risk-neutral." Under the probability measure specified by $\tilde{p}$ and $\tilde{q}$, the *expected* return of the risky asset equals the return on the risk free investment. In such a market, an investor who is neutral to risk, that is, who does not factor risk into investment decision, would be indifferent to investing in the stock and investing in the risk free instrument.

It must be emphasized that the risk-neutral probabilities are fictitious probabilities introduced for pricing derivatives. The "real" probabilities of up and down swings, if such exist, will in general be different. In fact, one should expect that for the real probabilities, $E[S_1] > e^{r\tau}$. The reason is that most investors are expected to be risk adverse. If the expected return on a risky asset is the same as for a risk-free instrument, investors will take the risk-free route and there will be little market for the risky asset. When $E[S_1] - e^{r\tau}$ is positive, this difference can be thought of as a premium necessary to induce investors to speculate on the risky asset.

Definition of the risk-neutral measure for the multi-period model.

Definition. In the multi-period, binomial tree model, the risk-neutral probability measure is the random walk probability model, as defined above, using $\tilde{p}$ for the probability of upswing and $\tilde{q} = 1 - \tilde{p}$ for the probability of downswing. That is, in each period, u occurs with probability $\tilde{p}$, d occurs with probability $\tilde{q}$, and market movements in different periods are independent.

Following the notation of section III of these notes, we should write expectation with respect to the risk-neutral measure as $E_{\tilde{p}}[\cdot]$. However, we will use instead the notation $\tilde{E}[\cdot]$.

The first important result partly justifies the terminology of "risk-neutral."

Basic fact: For any t,

$$S_0 = e^{-t(r\tau)}\tilde{E}\left[S_t\right]. \tag{12}$$

This generalizes (10) to any number of periods; $e^{-r\tau}$ is the discount factor per period, so $e^{-t(r\tau)}$ is the discount factor for r periods, with total duration of $t\tau$ years. Hence, under the risk-neutral measure, the discounted expected value of the price remains constant with the number of periods t.

Equation (12) is easy to derive. It is an immediate consequence of the formula (7) for expected value of the price using the random walk measure and of (11). Indeed, using these two formulas,

$$\tilde{E}[S_t] = S_0\left(\ell\tilde{q} + g\tilde{p}\right)^t = S_0(e^{r\tau})^t = S_0 e^{t(r\tau)},$$

and so (12) follows immediately.

We saw that in the one period model, (10) extends to a formula for the no-arbitrage price of any derivative: if V_1 is the derivative pay-off at the end of the first period

$$V_0 = e^{-r\tau}\tilde{E}[V_1].$$

This generalizes to the multi-period, binomial tree, as we state next.

C. Pricing using the risk-neutral measure.

Consider a derivative security whose underlying is the stock in our binomial tree model. Suppose the derivative expires at the end of period N, and denote its payoff by V_N. The backward induction algorithm we gave for pricing the derivative goes as follows. The algoritm actually gives a price $V_t(w_1, \ldots, w_t)$ for each t, $0 \leq t \leq N$ and market history $(w_1, \ldots, w_t)$ up to period t. These are found by solving, first for $t = N-1$, then for $t = N-2$, etcetera, the following equation:

$$V_t(w_1, \ldots, w_t) = e^{-r\tau}\left[\tilde{q}V_t(w_1, \ldots, w_t, d) + \tilde{p}V_t(w_1, \ldots, w_t, u)\right] \tag{13}$$

The following result is so important that we state it as a theorem.

Theorem 1 *Let V_0 be the price at time $t = 0$ found by solving (13). Then*

$$V_0 = e^{-N(r\tau)} \tilde{E}[V_N] \tag{14}$$

The no-arbitrage price of any derivative is also the discounted expected value of its payoff under the risk-neutral measure!

Example. We consider the problem solved in exercise 33, which is problem 1 on page 51 of the text. The solution is available online There the problem was to compute the price of a call option that expires at the end of the third period with strike 115. In the problem $S_0 = 120$, $g = 1.7$, $d = 0.8$ and the interest rate per period is 1.06 ($e^{r\tau} = 1.06$). In this case, we find

$$\tilde{p} = \frac{13}{45} \qquad \text{and} \qquad \tilde{q} = \frac{32}{45}.$$

Examining the price tree, the possible derivative payoffs are

$$\begin{aligned} 589.56 - 115 &= 474.56 \quad \text{and} \quad \tilde{I\!P}(V = 474.56) = \tilde{p}^3; \\ 277.44 - 115 &= 162.44 \quad \text{and} \quad \tilde{I\!P}(V = 162.44) = 3\tilde{p}^2\tilde{q}; \\ 130.56 - 115 &= 15.56 \quad \text{and} \quad \tilde{I\!P}(V = 15.56) = 3\tilde{p}\tilde{q}^2; \\ & \quad 0 \quad \text{otherwise.} \end{aligned}$$

because the first possibility occurs if there are three ups, the second if there are two ups and a down, and the third if there are two downs and an up. If there are three downs, the stock price falls below \$115 and the option expires worthless. Thus, according to Theorem 1,

$$V_0 = \frac{1}{1.06^3}\left[(15.56)3(13/45)(32/45)^2 + (162.44)3(13/45)^2(32/45) + (474.56)(13/45)^3\right] = 39.61$$

This is what we got before.

As a second example, study the text on page 59 and 60 where Theorem 1 is applied to price a look back option.

For small trees, applying Theorem 1 directly often gives a quicker procedure for calculating the option price.