640:495 Mathematical Finance, Problems

Ito's rule. In class we gave a prescription called Ito's rule for finding $d[f(X_t, t)]$ if $dX_t = \alpha_t\,dt + \beta_t\,dB_t$, where B is a standard Brownian motion. This rule had to do with using Taylor polynomial approximations, and replacing $(dB_t)^2$ by dt and dropping terms with $dtdB_t$ or (dt). By applying that prescription to a general f, Itô's rule can be stated generally, and in problems it is generally easier to work directly with these general statments than repeat the whole analysis each time. So we state the formulae here:

• For $f(x)$ a function of x only, having two continuous derivatives in x, and for $dX_t = \alpha_t\,dt + \beta_t\,dB_t$:

$$d[f(X_t)] = \left\{ f'(X_t)\alpha_t + \frac{1}{2} f''(X_t)\beta_t^2 \right\} dt + f'(X_t)\beta_t\,dB_t. \tag{1}$$

• For $f(x,t)$ having two continuous derivatives in x and one continuous derivative in t and for $dX_t = \alpha_t\,dt + \beta_t\,dB_t$:

$$d[f(X_t,t)] = \left\{ \frac{\partial f}{\partial t}(X_t,t) + \frac{\partial f}{\partial x}(X_t,t)\alpha_t + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}(X_t,t)\beta_t^2 \right\} dt + \frac{\partial f}{\partial x}(X_t,t)\beta_t\,dB_t. \tag{2}$$

The terms with second derivatives in f in the coefficients of dt in these formulae are called the "Itô correction terms." It is convenient to use the notations $f_t(x,t)$, $f_x(x,t)$ and $f_{xx}(x,t)$ for the partial derivatives of f.

Example: Find the Itô differential $d[B_t^2]$.

To do this apply formula (1) with $f(x) = x^2$ and $X_t = B_t$. Since $dX_t = dB_t$, $\alpha_t = 0$ and $\beta_t = 1$, Since $f'(x) = 2x$ and $f''(x) = 2$,

$$d[B_t^2] = \frac{1}{2} f''(B_t)\,dt + f'(B_t)\beta_t\,dB_t = dt + 2B_t\,dB_t.$$

Example: Find the Itô differential of tB_t^2. Here $f(x,t) = tx^2$, so $f_t(x,t) = x^2$, $f_x(x,t) = 2tx$, $f_{xx}(x,t) = 2t$. Hence

$$d[tB_t^2] = \{B_t^2 + t\}\,dt + 2tB_t\,dB_t.$$

In this problem set B_t always stands for standard Brownian motion.

78. Apply Itô's rule to compute (a) $d[B_t^3]$; (b) $d[B_t^4]$.

79. Let $Z_t = e^{B_t}$. Show that Z satisfies $dZ_t = (1/2)Z_t\,dt + Z_t\,dB_t$.

80. Let $Z_t = te^{B_t}$. Show that Z satisfies $dZ_t = [(1/2)Z_t + (1/t)Z_t]\,dt + Z_t\,dB_t$.

81. Let $f(x,t) = (x+t)e^{-x-(1/2)t}$. Show $f_t + (1/2)f_{xx} = 0$. Use Itô's rule to show that $d[(B_t+t)\exp\{-B_t - (1/2)t\} = (1-B_t-t)\exp\{-B_t - (1/2)t\}\,dB_t$.

In the next few problems we will use the fact, stated in the class notes, that if $E[\int_0^t \beta_t^2\,dt] = \int_0^t E[\beta_t^2]\,dt < \infty$, and if, for every $s < t$, $B_t - B_s$ and $\{\beta_u, u \le t\}$ are independent, then

$$E[\int_0^t \beta_s\,dB_s] = 0. \tag{3}$$

We cannot give a full proof of this fact at the level we are developing the subject But it is easily understood. Remember that we can think of $\beta_t dB_t$ as $\beta_t[B_{t+dt} - B_t]$ and can interpret this as the earnings on a bet of β_t on the fluctuation $dB_t = B_{t+dt} - B_t$. As β_t is independent of $dB_t = B_{t+dt} - B_t$, $E[\beta_t dB_t] = E[\beta_t]E[B_{t+dt} - B_t] = 0$. Now the integral $\int_0^t \beta_t\,dB_t$ is a essentially a sum of the terms $\beta_t\,dB_t$ and as the expectation of a sum is a sum of expectations, one arrives at (3). The condition $E[\int_0^t \beta_t^2\,dt] < \infty$ might look mysterious as it puts a bound on the expected square of the integrand; but it is used in the rigorous theory for making sure the integral is defined.

Here is a simple application. We know that $d[B_t^2] = 2B_t\,dB_t + dt$, which means $B_t^2 = \int_0^t 2B_s\,dB_s + t$. So taking expectations, $E[B_t^2] = t$. Of course, we know this already, and really it is used in showing $B_t^2 = \int_0^t 2B_s\,dB_s + t$. But at least this double checks (3).

Properly speaking we should always check the necessary condition $E[\int_0^t \beta_t^2\,dt] < \infty$, but in the problems you may assume this is true and use (3) as a tool to get answers.

82. Use the result of problem 78 (b) to show that $E[B_t^4] = 3t^2$ and $\mathrm{Var}(B_t^2) = 2t^2$.

83. Let $S_t = S_0\exp\{(\mu - \sigma^2/2)t + \sigma B_t\}$. We proved earlier that $E[S_t] = S_0 e^{\mu t}$. The purpose of this problem is to rederive the result using Itô's rule and (3). In class, we showed $dS_t = \mu S_t + \sigma S_t\,dB_t$ or $S_t - S_0 = \int_0^t \mu S_s\,ds + \int_0^t \sigma S_s\,dB_s$. Take expectations on both sides and interchange expectation and the ordinary integral. Then differentiate to obtain a differential equation for $y(t) = E[S_t]$. Show that $S_0 e^{\mu t}$ is the solution.

84. Let $V(S_T, T)$ be the payoff of a European option at time T. This could be a call option, in which case $V(S_T, T) = (S_T - X)^+$ or a put option, in which case $V(S_T, T) = (X - S_T)^+$. But it could be any other function of the price S_T. Assume that S follows the risk-neutral Black-Scholes model, $S_t = rS_t\,dt + \sigma S_t\,dB_t$.

Show the following. If $v(x,t)$ is a solution to

$$\begin{aligned} v_t(s,t) + rsv_s(s,t) + (\sigma^2/2)s^2 v_{ss}(s,t) - rv(s,t) &= 0, \qquad s > 0,\ 0 \le t < T; \\ v(s,T) = V(s,T), \quad s > 0, & \end{aligned}$$

then

$$v(S_0, 0) = e^{-rT} E\left[V(S_T, T)\right].$$

Hint: Apply Ito's rule to $e^{-rt}v(S_t, t)$ to derive an expression of the form $e^{-rT}V(S_T, T) = e^{-rT}v(S_T, T) = v(S_0, 0) + \int_0^T \beta_s \, dB_s$.

This problem is very important. It connects the Black-Scholes pde for option pricing to the formula for the price as an expectation with respect to the risk-neutral price, for general payoffs depending only the asset price at expiration.

---

85. (a) In this part you may use the fact, which you may assume without proof, that, if $Z \sim N(0, \sigma^2)$, then $E[Z^4] = 3\sigma^4$. The problem is to show that $Z^2 - \sigma^2$ has mean zero and variance $2\sigma^4$.

(b) In this part, you are asked to show what you used in (a): If $Z \sim N(0, \sigma^2)$, show that $E[Z^4] = 3\sigma^4$.

Hint: $E[Z^4] = \displaystyle\int_{-\infty}^{\infty} z^4 e^{-z^2/2\sigma^2} \frac{dz}{\sqrt{2\pi\sigma^2}}$. First change variables to replace z^2/σ^2 by y:

$$E[Z^4] = \sigma^4 \int_{-\infty}^{\infty} y^4 e^{-y^2/2\sigma^2} \frac{dz}{\sqrt{2\pi}}.$$

Notice that $y^4 e^[- y^2/2] = -y^3 \dfrac{d}{dy} e^{-y^2/2}$ and use this to integrate by parts. Use the same trick to integrate by parts again.

86. (a) Let B be a Brownian motion and let $0 = t_0 < t_1 < t_2 < \cdots t_n = t$ be a partition of $[0, t]$ into n equal subintervals, each of length t/n. (Thus $t_{i+1} - t_i = t/n$ for each i.) Consider

$$Y_n = \sum_{i=0}^{n-1} \left[B_{t_{i+1}} - B_{t_i}\right]^2 - t = \sum_{i=0}^{n-1} \left(\left[B_{t_{i+1}} - B_{t_i}\right]^2 - \frac{t}{n}\right).$$

Show that $E[Y_n] = 0$ and $\mathrm{Var}(Y_n) = \sum_0^{n-1} 2(t/n)^2 = 2t^2/n$.

(Hint: The terms in the sum defining Y_n are independent; explain why. Compute the variance of each term using the result of problem 85 (a) or problem 82. Remember that the variance of a sum of independent random variables is the sum of the variances.)

(b) Use the result of (a) to show

$$\lim_{t \to \infty} E\left[\left(\sum_{i=0}^{n-1} \left[B_{t_{i+1}} - B_{t_i}\right]^2 - t\right)^2\right] = 0.$$

(Remember that we have defined the limit of $\sum_{i=0}^{n-1}\left[B_{t_{i+1}}-B_{t_i}\right]^2$ as the quadratic variation of B over $[0,t]$. So with this problem we have proven that the quadratic variation of a Brownian motion over $[0,t]$ is t.)

87. Let B be a Brownian motion. Let $0=t_0<t_1<t_2<\cdots t_n=t$ be a partition of $[0,t]$ into n equal subintervals. Observe that ($B_0=0$

$$B_t^3 = B_t^3 - B_0^3 = \sum_{i=0}^{n-1}\left[B_{t_{i+1}}^3 - B_{t_i}^3\right].$$

a) By writing $B_{t_{i+1}} = B_{t_i} + (B_{t_{i+1}} - B_{t_i})$, show

$$B_{t_{i+1}}^3 - B_{t_i}^3 = 3B_{t_i}^2(B_{t_{i+1}} - B_{t_i}) + 3B_{t_i}^2(t_{i+1}-t_i) + 3B_{t_i}\left[(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1}-t_i)\right].$$

b) From (a), and the problem statement

$$B_t^3 = \sum_{i=0}^{n-1} 3B_{t_i}^2(B_{t_{i+1}} - B_{t_i}) + \sum_{i=0}^{n-1} 3B_{t_i}^2(t_{i+1}-t_i) + Y_n, \tag{4}$$

where $Y_n = \sum_{i=0}^{n-1} 3B_{t_i}^2\left[(B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1}-t_i)\right]$. As $n\to\infty$ the first two terms in (4) converge respectively to $\int_0^t 3B_s^2\,dB_s$ and $\int_0^t 3B_s^2\,ds$ by definition of the stochastic and Riemann integrals. In this part you are to show

$$E[Y_n^2] = \sum_0^{n-1} 6t_i(t/n)^2 \qquad \text{and} \qquad \lim_{n\to\infty} E[Y_n^2] \to 0.$$

It will then follow that $Y_n \to 0$ in mean square and so taking limits in (4), $B_t^3 = \int_0^t 3B_s^2\,dB_s + \int_0^t 3B_s^2\,ds$. This proves directly what we got by Itô's rule in problem 78 (a).

(Hint: For notational covenience, let $X_i = (B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1}-t_i)$. If you have done the previous problem you know this has mean 0 and variance $2t/n$. If not, show this using problem 85 (a) or problem 82. Then use independence of increments in computing $E[Y_n^2]$.)