640:495 Mathematical Finance, Problems

88. Consider an option that pays \$1 if $S_T > X$ and 0 otherwise, where S is the price of an underlying stock. This is the cash or nothing option. For notation, let

$$\mathbf{1}_{(X,\infty)}(x) = \begin{cases} 1, & \text{if } x > X; \\ 0, \text{otherwise}, & \end{cases}$$

denote its payoff. According to the Black-Scholes theory, if the underlying stock follows a Black-Scholes model with volatility σ and if the risk-free interest rate is r, then the price of the option is $w(S_t, t)$ where w solves

$$\begin{aligned} w_t(s,t) + \frac{1}{2}\sigma^2 s^2 w_{ss}(s,t) + rsw(s,t) - rw(s,t) &= 0 \\ w(s,T) = \mathbf{1}_{(X,\infty)}(s) & \end{aligned}$$

Show that

$$W(s,t) = e^{-r(T-t)} N(d_1(s,t)),$$

where

$$d_1(s,t) = \frac{\ln(s/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

is a solution to the Black-Scholes pde and hence $e^{-r(T-t)}N(d_1(S_t,t))$ gives the price of the option. This calculation is a bit involved, but not hard

(b) How did we guess the solution of part (b)? We get it from the risk-neutral measure approach. This approach says the option price should be

$$W(t) = E\left[\mathbf{1}_{(X,\infty)}(S_T) \mid S_u, u \le t\right].$$

Compute this expectation directly, verifying that indeed $W(t) = w(S_t, t)$.

Hint: For a discussion providing some background and hints, see section 6.5.1 of the text.

89. The Cox-Ingersoll-Ross price model supposes

$$dS_t = rS_t\,dt + \sigma\sqrt{S_t}\,dB_t.$$

In general, one cannot write down an explicit formula for a process satisfying this equation, but one can show that such a process does exist. Suppose we are trying to price an derivative that pays $U(S_T)$ at expiration. Following the argument used to derive the Black-Scholes pde, derive a pde for a function $u(s,t)$ that prices the derivative as $u(S_t, t)$. The only change is that $dS_t = rS_t\,dt + \sigma S_t\,dB_t$ instead of the Black-Scholes model, and you need to take this into account when applying Itô's rule.

90. Do problems 2 and 3 on pages 119 and 120 of the text.

91. Do problems 1(a)(b) and 2(a)(b) on page 133 of the text.

92. The Asian option. The payoff of an Asian option with strike X and expiration T is

$$V(T) = \max\{\frac{1}{T}\int_0^T S_u\,du - X, 0\}.$$

Our goal is to price this using the pde approach, assuming the risk-netural Black-Scholes price model for the underlying asset:

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t 5.$$

The strategy is the same that is applied in the lecture notes. Can we find a portfolio Π_t specified by an investment strategy ϕ_t, so that $d\Pi_t = (\Pi_t - \phi_t S_t)r\,dt + \phi_t\,dS_t$, so that $\Pi_T = V(T)$? If we can, then no-arbitrage says that the price of the option should be precisely Π_t at all times t.

For options whose payoffs were simply functions of S_T we carried out this strategy by looking for a replicating portfolio whose value had the form $\Pi_t = v(S_t, t)$. We cannot hope that this will works for the Asian option since its payoff is path dependent, and so its price cannot depend on S_t alone. Instead, the trick will be to introduce a new process

$$Y_t = \int_0^t S_u\,du,$$

and to look for a replicating portfolio of the form

$$\Pi_t = v(S_t, Y(t), t),$$

where $v(s, y, t)$ is now a function of 3 variables. In order to have $\Pi_T = v(S_T, Y_T, T)$ be the payoff of the Asian option, we need

$$v(s, y, T) = \max\{\frac{y}{T} - X, 0\}.$$

This will be the terminal condtion

To derive a pde for v, we need to know how to calculate the stochastic differential

$$dv(S_t, Y_t, t).$$

Because $Y(t)$ is differentiable in t with derivative $Y_t' = S_t$, we see that, using the chain rule of multivariable calculus

$$\frac{\partial}{\partial t} g(s, Y(t), t) = g_y(s, Y_t, t)Y_t' + g_t(s, Y_t, t) = g_y(s, Y_t, t)S_t + g_t(s, Y_t, t).$$

Thus, to calculate the stochastic differential one should apply Itô's rule as if to $g(S_t, t)$ but replace the term $g_t(S_t, t)$ by $g_y(s, Y_t, t)S_t + g_t(s, Y_t, t)$:

$$
\begin{aligned}
dg(S_t, Y_t, t) &= \left\{ g_y(S_t, Y_t, t)S_t + g_t(S_t, Y_t, t) + rS_t g_s(S_t, Y_t) + \frac{1}{2}(\sigma S_t)^2 g_{ss}(S_t, Y_t, t) \right\} dt \\
&\quad + g_s(S_t, Y_t, t)\sigma S_t \, dB_t.
\end{aligned}
$$

Following the steps deriving the Black-Scholes pde in the lecture notes, find a pde for $v(s, y, t)$ and also determine what ϕ_t should be for the replicating portfolio.

93. (a) We found that for a European call option, the delta at time t, price S_t, risk free rate r, volatility σ, expiration T, and strike X is $N(d_1(S_t, t))$ where $d_1(s, t) = (\ln(s/X) + (r + \sigma^2/2)(T - t))/(\sigma\sqrt{T - t})$. Find the delta of a European put with the same data. Show that the delta for the put is always negative.

(b) Calculate also, gamma and theta for the put option.

94. (a) Suppose that you sell 1000 call options. The calls expire in 90 days. The risk free interest rate is 5%, today's stock price is 60, the strike price is 60 and the volatility is 1. Calculate the value $V(60, 0)$ of the option today ($t = 0$) Suppose the stock price rises to 61 the next day. Calculate the value $V(61, 1/365)$ option at this point. Since you have sold this option, you lose or gain $-1000(V(61, 1/365) - V(60, 0))$. Calculate this amount. It is the loss or gain on an unhedged (*naked*) position of being short the 1000 calls.

(b) Create a delta-neutral portfolio by purchasing or short selling the stock on which the call options are written. This portfolio hedges the call option position. How many shares should you buy or short sell? Now suppose the price goes up the next day to 61. Calculate the loss or gain of the hedged position.

(c) Compare the loss or gain on the hedged and unhedged positions if the price declines to 59.

95. In the previous problem, find the price S on day one which maximizes the value of the hedged portfolio.

96. Suppose you have written 1000 put options on the same stock as in problem 94. Assume the strike is again 60. Create a delta neutral portfolio and compute its value after one day if the price rises to 63 and also if the price falls to 58. (Use the answer to 93(a)).

97. Show that a delta-neutral portfolio hedging a single call option will gain in value with time if the stock price remains constant, as well as r and σ, and if the value of the delta-neutral portfolio is initially zero. (Translate this word description into a statement about the greek Θ_t at $t = 0$. Use the Black-Scholes pde to draw your conclusion.)

Exercises 94, 95, 96, and 97 are inspired by the excellent discussion in Chapter 9 of Capinski and Zastawniak, *Mathematics for Finance*, Springer-Verlag, 2003.

98. (a) Let $C^{T,X}(S,t)$ be the price of a call expiring at T and strike X. An-at-the-money call is a call whose strike is the current stock price ($S_t = X$). Show that, at the money,

$$\Gamma_{C^{T,X}}(X) = \frac{1}{\sigma X\sqrt{2\pi(T-t)}} \exp\{-(r-\sigma^2/2)^2(T-t)/\sigma^2\}.$$

(b) Suppose you have sold 1000 calls on a stock that expires in 4 months at a strike of 70. Assume $r = 0.055$ and $\sigma = 0.40$. Let the value of this call be $C_1(S)$, as a function of the price today. Calculate Γ for this $S = 80$, and time and call this Γ_1. Consider an at-the-money ($X = 80$) call on the same stock that expires in one=half month's time, calling its price $C_2(S)$. Compute its Γ, call it Γ_2, when $S = 80$. Create a portfolio $\Pi(S)$ in x shares of stock short the 1000 calls expiring in 4 months, and with y of the second, at-the-money calls expiring in one-half month, such that the delta and the gamma of $\Pi(S)$ are both zero. (Determine x and y). This portfolio is delta-gamm neutral and protects well against the risk of large price swings.