Financial Mathematics, 640:495: Hedging and portfolio replication in the binomial tree model.

These notes complement section 3.7 of the text. We start with our binomial tree model of N periods. We keep the usual notation: S_0 is the asset price at time $t{=}0$; r is the risk-free rate, and $e^{-r\tau}$ is the per period discount factor. Prices move according to

$$S_{t+1} = \begin{cases} gS_t & \text{if } u \text{ occurs in period } t{+}1; \\ \ell S_t & \text{if } d \text{ occurs in period } t{+}1; \end{cases} \tag{1}$$

The no arbitrage condition $\ell < e^{r\tau} < g$ holds, and $\tilde{p} = (e^{r\tau} - \ell)/(g - \ell)$, $\tilde{q} = 1 - \tilde{p}$.

In the multi-period model we are allowed to rebalance the portfolio at the beginning of each time period. We will let δ_t be the number of shares of stock we choose to hold in period $t + 1$. Of course, we may let δ_t depend on the market history up to time t; we write this as $\delta_t = \delta_t(w_1, \ldots, w_t)$ to indicate this dependence. Suppose that we have a portfolio whose dollar value at time t is Π_t (of course $\Pi_t = \Pi_t(w_1, \ldots, w_t)$ also. Then for period $t{+}1$, we hold δ_t shares of stock, which costs $\delta_t S_t$ and invest the rest, which is $\Pi_t - \delta_t S_t$ dollars, at the risk free rate. Therefore at time $t{+}1$, the value of the portfolio is

$$\Pi_{t+1} = e^{r\tau} (\Pi_t - \delta_t S_t) + \delta_t S_{t+1} \tag{2}$$

This is the portfolio update equation. It assumes that there are no additional funds that come into the investor from outside during period $t{+}1$, and for this reason, portfolio processes obeying (2) are called **self-financing.** Of course, such a portfolio process will start with some initial endowment of money P_0. Once the initial endowment Π_0 and the investment decisions $\delta_0, \delta_1, \ldots, \delta_{N-1}$ are set, (2) determines all the remaining portfolio values $\Pi_1, \ldots, \Pi_N$

Example. Let $S_0 = 100$, $g = 1.1$ $\ell = 0.9$, $e^{r\tau} = 1.02$. and suppose $\Pi_0 = \$500$. $\delta_0 = 3$, $\delta_1(u) = 2$, $\delta_1(d) = 4$. We shall calculate the portfolio process along all market histories. Using (2)

$$\begin{aligned} \Pi_1(u) &= e^{r\tau}(500 - 3S_0) + 3(1.1)(100) = (1.02)200 + 330 = 534 \\ \Pi_1(d) &= e^{r\tau}(500 - 3S_0) + 3(0.9)(100) = (1.02)200 + 270 = 474 \\ \Pi_2(u,u) &= e^{r\tau}(534 - \delta_1(u)S_1(u)) + \delta_1(u)S_2(u,u) \\ &= (1.02)(534 - 2(110)) + 2(1.1)^2(100) = 562.28 \end{aligned}$$

$$\begin{aligned}
\Pi_2(u,d) &= e^{r\tau}(534 - \delta_1(u)S_1(u)) + \delta_1(u)S_2(u,d) \\
&= (1.02)(534 - 2(110)) + 2(1.1)(.9)(100) = 518.28 \\
\Pi_2(d,u) &= e^{r\tau}(474 - \delta_1(d)S_1(u)) + \delta_1(d)S_2(d,u) \\
&= (1.02)(474 - 4(90)) + 4(1.1)(.9)(100) = 512.28 \\
\Pi_2(d,d) &= e^{r\tau}(474 - \delta_1(d)S_1(u)) + \delta_1(d)S_2(d,u) \\
&= (1.02)(474 - 4(90)) + 4(.9)^2(100) = 440.28
\end{aligned}$$

Notice that although $S_2(u,d) = S_2(d,u)$, $\Pi_2(u,d) \neq \Pi_2(d,u)$, because different portfolio choices were made for period 2 depending on whether the market went up or down in the first period.

Let V_N be a contingent claim. That is V_N is a payoff at time N. A portfolio process determined by Π_0,$\delta_0, \ldots, \delta_{N-1}$ is said to replicate V_N if

$$V_N(w_1, \ldots, w_N) = \Pi_N(w_1, \ldots, w_N) \quad \text{for all market outcomes } w_1, \ldots, W_N.$$

We will show how to replicate any contingent claim, using the delta hedging formula for the one period model. This is important because we are showing any contingent claim in a binomial tree model can be replicated if trading is allowed in each period.

The idea is simple: first we determine the prices of the derivative for all times and market histories using backward induction. To do this, recall that we recursively compute $V_{N-1}, V_{N-2}, \ldots, V_0$ by the equation

$$V_t(w_1, \ldots, w_t) = \frac{1}{e^{r\tau}} \left[V_{t+1}(w_1, \ldots, w_t, d)\tilde{q} + V_{t+1}(w_1, \ldots, w_t, u)\tilde{p} \right] \qquad (3)$$

Now let us remember what delta hedging says if we apply it to the situation of this equation. In the notation of the one period model the role of U is played by $V_{t+1}(w_1, \ldots, w_t, u)$ and that of D by $V_{t+1}(w_1, \ldots, w_t, d)$ Therefore we define

$$\triangle_t(w_1, \ldots, w_t) = \frac{V_{t+1}(w_1, \ldots, w_t, u) - V_{t+1}(w_1, \ldots, w_t, d)}{S_{t+1}(w_1, \ldots, w_t, u) - S_{t+1}(w_1, \ldots, w_t, d)} \qquad (4)$$

The one-period delta hedging result says that if we start with $V_t(w_1, \ldots, w_t)$ in cash, buy $\triangle_t(w_1, \ldots, w_t)$ shares of stock and invest the rest at the risk free rate, we will replicate the payoffs $V_{t+1}(w_1, \ldots, w_t, u)$ if the market goes up in period $t+1$ and $V_{t+1}(w_1, \ldots, w_t, d)$ if the market goes down.

From this we derive the replicating portfolio. It is:
start with $\Pi_0 = V_0$ and follow the strategy $\delta_t = \triangle_t$ for all $t = 0, 1, \ldots, N-1$.

To see why this works, notice first that if $\Pi_0 = V_0$ and one uses $\triangle_0$, then the portfolio replicates $V_1(w_1)$ in the first period. That means that $\Pi_1(u) = V_1(u)$ and $\Pi_1(d) = V_1(d)$. Suppose u occurs in the first period, then we purchase $\delta_1(u)$ shares of stock to hold in period 2 and invest the rest $\Pi_1(u) - \triangle S_1(u) = V_1(u) - \triangle S_1(u)$ at the risk free rate and that duplicates $V_2(u,u)$ and $V_2(u,d)$; in other words, $\Pi_2(u,u) = V_2(u,u)$ and $\Pi_2(u,d) = V_2(u,d)$. Proceeding in this way, one sees that $\Pi_2 = V_2$, then $\Pi_3 = V_3$ and so on recursively until $\Pi_N = V_N$. This argument might seem a bit abstract, but it is just to apply one-period delta hedging period by period. Follow through the following example to see concretely why the procedure works.

Example. We consider the set up of the previous example. (g, ℓ, r, and S_0 are all the same.) The student should follow the example filling in the appropriate trees. Consider a call option at strike $97. Notice that $\tilde{p} = 3/5 = 0.6$

Then applying the call payoff and backward induction,

$$\begin{array}{rll} V_2(u,u) = 24, & V_2(u,d) = V_2(d,u) = 2, & V_2(d,d) = 0 \\ V_1(u) = 76/5.1, & V_1(d) = 6/5.1, & V_0 = 240/[(1.02)25.5]. \end{array}$$

On the other hand

$$\begin{aligned} \triangle_0 &= \frac{V_1(u) - V_1(d)}{S_1(u) - S_1(d)} = \frac{70/5.1}{20} = \frac{7}{10.2} \\ \triangle_1(u) &= \frac{V_2(u,u) - V_1(u,d)}{S_2(u,u) - S_2(u,d)} = \frac{22}{22} = 1 \\ \triangle_1(d) &= \frac{V_2(d,u) - V_1(d,d)}{S_2(d,u) - S_2(d,d)} = \frac{2}{18} \end{aligned}$$

Now let us check replication. Set $\Pi_0 = V_0 = 240/25.5$. Then buy $\delta_0 = 7/10.2$ shares of stock. If u occurs

$$\begin{aligned} \Pi_1(u) &= (1.02)(V_0 - \triangle_0(100)) + \triangle_0(110) \\ &= (1.02)\left(\frac{240}{(1.02)25.5} - \frac{700}{1.02}\right) + \frac{770}{10.2} = \frac{76}{5.1} = V_1(u). \end{aligned}$$

If d occurs

$$\begin{aligned}\Pi_1(d) &= (1.02)(V_0 - \triangle_0(100)) + \triangle_0(90) \\ &= (1.02)\left(\frac{240}{(1.02)25.5} - \frac{700}{1.02}\right) + \frac{720}{10.2} = \frac{6}{5.1} = V_1(d).\end{aligned}$$

Given u in the first period, apply the delta hedge with $\triangle(u) = 1$. Then

$$\Pi_2(u,u) = (1.02)(\frac{76}{5} - 110) + 121 = 24 = V_2(u,u).$$

The student should finish the remaining cases.