Financial Mathematics, 640:495: Normal Random Variables and the Central Limit Theorem

1. Continuous random variables and variance. A function f defined on the real line satisfying the two conditions

$$f(x) \geq 0 \text{ for all } x, \text{ and } \qquad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1, \tag{1}$$

is called a *probability density function.* If f is a probability density function and if X is a random variable such that

$$\mathbb{P}(X \leq z) = \int_{-\infty}^{z} f(x)\,\mathrm{d}x, \qquad \text{for all } z, \tag{2}$$

we say that X is a *continuous random variable* and that f is the probability density funtion of X. Given a continuous random variable X, we often denote its probability density function by f_X. It is a simple consequence of (2) that $\mathbb{P}(X = b) = 0$ for any individual b, and

$$\mathbb{P}(a < X \leq b) = \mathbb{P}(a \leq X \leq b) = \mathbb{P}(a < X < b) = \mathbb{P}(a \leq X < b) = \int_{a}^{b} f_X(x)\,\mathrm{d}x.$$

Expectation for continuous random variables is calculated according to

$$E\left[h(X)\right] = \int_{-\infty}^{\infty} h(x) f_X(x)\,\mathrm{d}x. \tag{3}$$

Recall that if X is a random variable with mean $E[X] = m$, its variance is defined by $\mathrm{Var}(X) = E[(X-m)^2]$ and it is often convenient to compute this using the identity $\mathrm{Var}(X) = E[X^2] - m^2$. We shall use often the following. Let X be a random variable with mean m and variance σ^2. then

$$aX + b \text{ is a random variable with mean } am + b \text{ and variance } a^2\sigma^2. \tag{4}$$

This is easy to see because $E[aX + b] = aE[X] + b = am + b$. and

$$\mathrm{Var}(aX+b) = E\left[(X+b-(am+b)^2\right] = E\left[(aX-am)^2\right] = a^2E[(X-m)^2] = a^2\sigma^2.$$

If X and Y are *independent* random variables with means m_X and m_Y respectively, then $E[(X-m_X)(Y-m_Y)] = E[X-m_X]E[Y-m_Y] = 0$. This fact is used to show that if $X_1, \ldots, X_n$ are independent, then

$$\mathrm{Var}\left(\sum_{1}^{n} X_i\right) = \sum_{1}^{n} \mathrm{Var}(X_i). \tag{5}$$

2. **Normal random variables.** A random variable X is said to have a nomal distribution with mean m and variance σ^2 if

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/(2\sigma^2)}. \tag{6}$$

We write this short hand as $X \sim N(m, \sigma^2)$.

If $X \sim N(0, 1)$, then X is said to be a standard normal random variable. The probability distribution function of the standard normal distribution function will appear in the Black-Scholes formula and it is denoted,

$$N(x) = \int_{-\infty}^{x} e^{-y^2/2} \frac{\mathrm{d}y}{\sqrt{2\pi}}, \qquad -\infty < x < \infty. \tag{7}$$

Note that the $\sqrt{2\pi}$ term appears in these formulas because of the identity,

$$\int_{-\infty}^{\infty} e^{-y^2/2} \frac{\mathrm{d}y}{\sqrt{2\pi}} = 1 \tag{8}$$

One may check that if $X \sim N(m, \sigma^2)$, then indeed $E[X] = m$ and $\mathrm{Var}(X) = \sigma^2$.

The following is a basic and useful fact:

$$\text{if } X \sim N(m, \sigma^2), \text{ then } aX + b \sim N(am+b, a^2\sigma^2). \tag{9}$$

We know from (4) that the mean of X is $am+b$ and its variance is $a^2\sigma^2$, so the interesting part of this statement is that $aX+b$ is normal if X is normal. To show that (9) is true one needs to show that $X \sim N(m, \sigma^2)$ implies,

$$I\!P\left(aX+b \le z\right) = \int_{-\infty}^{z} e^{-(x-(am+b))^2/2(a^2\sigma^2)} \frac{\mathrm{d}x}{\sqrt{2\pi a^2\sigma^2}}.$$

This is left to the student.

A particular case of (9) is: if $X \sim N(m, \sigma^2)$, then $(X-m)/\sigma \sim N(0, 1)$. This has many applications. As a first application, one can compute probabilities for any normal random variable using tables for the standard normal distribution N.

Example. Let $X \sim N(2, 9)$. Find $I\!P(-2 < X < 4)$. From what wwe have said, $(X-2)/3$ is standard normal. Thus,

$$\begin{aligned} I\!P(-2 < X < 4) &= I\!P\left(\frac{-2-2}{3} < \frac{X-2}{3} < \frac{4-2}{3}\right) \\ &= I\!P\left(-4/3 < (X-2)/3 < 2/3\right) = N(2/3) - N(-4/3). \end{aligned}$$

The standard normal density function is symmetric about zero and so $N(-4/3) = 1 - N(4/3)$. Thus, the answer can be written as $N(2/3) + N(4/3) - 1$. We do this because tables often present $N(x)$ only for $x > 0$. $\diamond$

3. An important expected value calculation for normals. We will show the following

$$\text{if } X \sim N(m, \sigma^2), \text{ then } \quad E[e^{\lambda X}] = e^{\lambda m + (\sigma^2 \lambda^2/2)}. \tag{10}$$

First assume that $Y \sim N(0, 1)$. Then the recipe (3) for computing expectations tells us,

$$E[e^{\lambda X}] = \int_{-\infty}^{\infty} e^{\lambda x} e^{-x^2/2} \, \frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

Completion of the square in the exponent of the integrand gives,

$$e^{\lambda x} e^{-x^2/2} = e^{\lambda x - x^2/2} = e^{\lambda^2/2} e^{-(x-\lambda)^2/2}.$$

Thus

$$E[e^{\lambda X}] = e^{\lambda^2/2} \int_{-\infty}^{\infty} e^{-(x-\lambda)^2/2} \, \frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

However, by the change of variables $z = (x - \lambda)$, and by (8),

$$\int_{-\infty}^{\infty} e^{-(x-\lambda)^2/2} \, \frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} e^{-z^2/2} \, \frac{\mathrm{d}z}{\sqrt{2\pi}} = 1.$$

So we obtain,

$$\text{if } X \sim N(0, 1), \text{ then } \quad E[e^{\lambda X}] = e^{\lambda^2/2}. \tag{11}$$

Now consider $X \sim N(m, \sigma^2)$. Observe that

$$\lambda X = \lambda(\sigma \left(\frac{X - m}{\sigma} + m \right) = \lambda\sigma \frac{X - m}{\sigma} + \lambda m.$$

Thus, $\quad E\left[e^{\lambda X}\right] = e^{\lambda m} E\left[e^{\lambda\sigma(X-m)/\sigma}\right]$. But $(X - m)/\sigma$ is standard normal, so we can apply (11) to the last term with $\lambda\sigma$ in place of λ. The result is

$$E[e^{\lambda X}] = e^{\lambda m} e^{\sigma^2 \lambda^2/2},$$

which is the same as formula (10) that we wanted to show. $\diamond$

4. The Central Limit Theorem. The reason that the normal distribution is important is the Central Limit Theorem, explained in this section.

Consider a sequence of independent random variables, $\xi_1, \xi_2, \xi_3, \ldots$ all with the same distribution. Such sequences appear throughout probability and are important in statistical models as well. For example, independent, identically distributed random variables model repetitions of a random trial or successive random samples in statistics. In finance, the market movements in successive periods under the risk-neutral measure are independent and identically distributed.

The Central Limit Theorem, addresses the question of the probability distribution of $\sum_{i=1}^n \xi_i$, suitably rescaled and centered, as n increases toward infinity. Assume that the ξ_i have finite means and variances. As they are assumed all to have the same distribution, the mean and variance for each ξ_i is the same. Let $m = E[\xi_i]$ denote this common mean $m = E[\xi_i]$, and $\sigma^2 = \mathrm{Var}(\xi_i)$ this common variance. Our first objective is to transform the sum $\sum_{i=1}^n \xi_i$ by additive and multiplicative factors to get a random variable with mean 0 and variance 1. To do this, observe that

$$E\left[\sum_{i=1} \xi_i\right] = \sum_{i=1}^n E[\xi_i] = \sum_{i=1}^n m = nm,$$

and

$$\mathrm{Var}\left(\sum_{i=1} \xi_i\right) = \sum_{i=1} \mathrm{Var}(\xi_i) = n\sigma^2.$$

In this last equation, we used formula (5). According to (4),

$$\frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1} \xi_i - \sum_{i=1}^n m\right) = \frac{1}{\sigma\sqrt{n}}\sum_{i=1}(\xi_i - m) \qquad \text{has variance 1.}$$

Theorem 1 The Central Limit Theorem. *Assume $\xi_1, \xi_2, \xi_3, \ldots$ are independent and identically distributed with common mean m and common variance σ^2. Then*

$$\lim_{n\to\infty} I\!\!P\left(\frac{1}{\sigma\sqrt{n}}\sum_{i=1}(\xi_i - m) \le z\right) = N(z) \quad \textit{for all } z. \tag{12}$$

Since $N(z)$ is the probability that a standard normal random variable is less than or equal to z, the Central Limit Theorem says that, no matter what the distribution of the ξ_i's is, so long as their mean is m and their variance is σ^2, the distribution of $\frac{1}{\sigma\sqrt{n}}\sum_{i=1}(\xi_i - m)$ is approximately standard normal for n large.

Note that if $\frac{1}{\sigma\sqrt{n}}\sum_{i=1}(\xi_i - m)$ is approximately standard normal, then

$$\frac{1}{\sqrt{n}}\sum_{i=1}(\xi_i - m) \quad \text{is approximately an } N(0,\sigma^2) \text{ r.v.} \tag{13}$$

The Central Limit Theorem can be used to approximate probabilities.

Example. Let $\xi_1, \xi_2, \ldots$ be independent, all with distribution $I\!P(\xi_i = 1) = 2/3$ and $I\!P(\xi_i = -1) = 1/3$. Then $m = 1/3$ and $\sigma^2 = 8/9$. Consider $X = \sum_1^{60} \xi_i$. If we think of $\xi_1, \xi_2, \ldots$ as being the successive steps of a random walk, X is the distance the random walk has moved from its starting point in 60 steps. Our problem is to compute, approximately, $I\!P(12 \le X \le 28)$. (This probability can be represented exactly in terms of probabilities of a binomial random variable, but the formula is long and hard to compute numerically.)

X is a discrete random variable and if we apply the Central Limit Theorem, we will be approximating it by a continuous random variable. Because if this, we will get more accurate results if we apply the so-called continuity correction. X is integer valued, so we shall work instead with $I\!P(11.5 < X < 28.5)$. Here 11.5 is chosen because it is halfway between 12 and the next possible value of X less than 12, namely 11; likewise 28.5 is midway between 28 and the next possible value of X greater than 28. Observe that the events $11.5 < X < 29.5$ and $12 \le X \le 28$ are the same. By the Central Limit Theorem,

$$\frac{1}{\sqrt{8/9}\sqrt{60}}(X - (1/3)60) = \frac{\sqrt{3}}{4\sqrt{10}}(X - 20)$$

should be approximately standard normal. Thus

$$\begin{aligned} I\!P(11.5 < X < 28.5) &= I\!P\left(\frac{11.5-20}{(4/3)\sqrt{10}} < \frac{\sqrt{3}}{4\sqrt{10}}(X-20) < \frac{18.5-20}{(4/3)\sqrt{10}}\right) \\ &\approx N(2.016) - N(-2.016) = 0.956. \end{aligned}$$