Financial Mathematics, 640:495: Brownian motion and Samuelson's asset price model.

1. Modeling in continuous time and with a continuous state space. The time parameter in the binomial tree market models is discrete; trading is assumed to occur only at a fixed sequence of times $t = 0$, τ, 2τ, $\cdots$, $T = N\tau$, where T is the final trading time and the expiration date of any derivative that we are analyzing. In these models it is also assumed that the state space, by which we mean the range of possible market outcomes or prices, is discrete. Thus, it has been necessary only to model a finite sequence of prices, $S_0, S_1, \ldots, S_n$, where S_i denotes the price at the end of the i^{th} period, and this we did by specifying only three parameters, the initial price S_0, the factor, g, by which the price changes in upswings, and the factor, ℓ, by which it changes in downswings.

It is clearly more realistic to have models in which the time parameter can vary continuously from the starting time zero to the expiration date T, because trading on an asset can occur at any time during the day, and asset prices in a market, especially for heavily traded stocks, do fluctuate continuously. Moreover, although prices trade in increments of cents or higher, they have a potentially large range of closely spaced values, and it would make sense to allow prices to have any positive value. In the next part of the course, we will move to models in which S_t is defined for all t in the interval $[0, T]$, S_t can take any of a continuous range of values. The object is to price derivatives once the asset price model is specified.

The reference models for asset prices in continuous time will be random process models. That is, the underlying asset price will be a random process $\{S_t; 0 \le t \le T\}$, in which the time index t varies continuously in $[0, T]$. Formally, a random process $\{X_t; 0 \le t \le T\}$, is just a collection of random variables indexed by t. So, at every time t, the price S_t will be a random variable. But it will also be important conceptually to think of the price process in the following way, analogous to how we worked with binomial tree models. We let Ω denote the set of all market histories over the interval $[0, T]$ and for each market history ω, we let $\{S_t(\omega); t \in [0, T]\}$ denote the price path determined by ω. If we imagine that ω is random, we can think of the function of t given by $\{S_t(\cdot); t \in [0, T]\}$ as a *random path.* Notice that we have not given in this discussion an explicit mathematical representation of ω. (This could be done, but it would take us too far afield.) We will generally write S_t without the ω dependence. You should understand in the continuous-time context that S_t is always a random variable for fixed t and that $\{S_t; t \in [0,T]\}$ may be viewed as a random path.

How does one come up with continuous-time random processes models, whatever the field of application? One standard method, the one we shall in fact use, derives them as limits of discrete time models as the time increment between steps goes to zero.

2. Brownian motion. Physical Brownian motion is the fast and erratic movement, due to random molecular bombardment, of microscopic particles suspended in a fluid. It is named after the botanist Robert Brown who undertook the first careful experimental study of this phemomena in the 1820's. In 1905, Einstein derived the statistical properities of Brownian motion from idealized assumptions. Earlier, in 1900, Bachelier realized a similar theory in a model of stock market prices. The fully rigorous mathematical theory of Brownian motion was initiated by Norbert Wiener in the 1930's; for this reason the term *Wiener process* is sometimes used for Brownian motion. We shall use "Brownian motion" to refer to the idealized mathematical model of physical Brownian motion.

We shall first give the formal definition and then try to explain and motivate it.

Definition. A random process $\{B_t; t \geq 0\}$ is called a standard Brownian motion if $B_0 = 0$ with probability 1 and:

(i) the (random) paths of $\{B_t; t \geq 0\}$ are continuous in t;

(ii) (independent increments property) for each $0 \leq s < t$, $B_t - B_s$ is independent of $\{B_u, u \leq s\}$, the history of B up to time s;

(iii) (the stationary increments property) for each $u \geq 0$, $v \geq 0$, and $h > 0$, the probability distribution of $B_{v+h} - B_v$ is the same as the probability distribution of $B_{u+h} - B_u$.

(iv) For any $t \geq 0$, $B_t \sim N(0,t)$.

Remarks on the definition. 1. For $0 \leq s < t$, the difference $B_t - B_s$, which is the amount B changes by between times s and t, is called the increment of B on $[s,t]$. Thus, condition (ii) says that the increment of B on $[s,t]$ is independent of anything that has happened before time s; condition (iii) says that the probability distribution of the increment $B_{u+h} - B_u$ of B on $[u, u+h]$ depends only on the length of the interval h and not on the time u when the interval begins. Assumption (iii) implies in particular that the mean of B_t is zero for every t. If B were to represents one's fortune in a game of chance, the game is fair in the sense that one stands neither to gain nor lose on the average over any time interval.

2. Let $0 < s < t$. By combining (iii), with $v = s$, $v+h = t$, and $u = 0$, it follows that $B_t - B_s$ has the same probability distribution as $B_{t-s} - B_0 = B_{t-s}$ (since $B_0 = 0$). So, using (iv), $B_t - B_s \sim N(0, t-s)$. By (ii), $B_t - B_s$ and B_s are independent. In summary, if $0 < s < t$,

$$B_s \sim N(0, s), \quad B_t - B_s \sim N(0, t-s), \quad \text{and } B_t - B_s \text{ and } B_s \text{ are independent.} \tag{1}$$

Brownian motion is a continuous time analogue of random walk. Intuitively, it can be derived as a continuous time limit of random walks in which, simultaneously, the time between steps and the step size go to zero. To express this mathematically, let $\xi_1, \xi_2, \ldots$ be independent random variables, each having the same probability distribution with mean 0 and variance 1. We will define a sequence of random walks $\{B^N\}$ that take steps more and more frequently. The first walk in the sequence is

$$B^1_t = \begin{cases} 0, & \text{if } 0 \le t < 1; \\ \sum_{i=1}^k \xi_i, & \text{if } k \le t < k+1. \end{cases}$$

Thus $B^1_t = \xi_1$ if $1 \le t < 2$, $B^1_t = \xi_1 + \xi_2$ if $2 \le t < 3$, and so on. $\{B^1_t, t \ge 0\}$ is a random walk which takes steps at integer times and the k^{th} step has value ξ_k. We can think of B^1_n as representing how much an asset price changes between time $t = 0$ and time $t = n$ in a model that allows trading only at integer times. The assumption that the steps ξ_i are independent means that each new price movement is independent of the previous price history, and this is the version of the independent increments property for discrete time steps. The assumption that all the steps have the same probability distribution implies that increments of B^1_t on time intervals $[n, n+k]$ are stationary, that is, their distributions depend on k only, not on n. Finally, the steps have zero mean so the walk will fluctuate around its starting point at 0.

Now we want to speed up the time between steps. At the same time, we must decrease the step size, so the walk does not fluctuate too wildly. As a convention, let us choose the step size so that the variance of the position at time $t = 1$ is always equal to 1. To illustrate, we define $\{B_t^2; t \geq 0\}$, which takes steps at time intervals of length $1/2$. This is:

$$B_t^2 = \begin{cases} 0, & \text{if } 0 \leq t < 1/2; \\ \frac{1}{\sqrt{2}} \sum_{i=1}^{k} \xi_i, & \text{if } k/2 \leq t < (k+1)/2. \end{cases}$$

Thus $B_t^2 = \xi_1/\sqrt{2}$ if $1/2 \leq t < 1$, $B_t^2 = (\xi_1 + \xi_2)\sqrt{2}$ if $2 \leq t < 3$, and so on. Observe that

$$\mathrm{Var}(B_1^2) = \mathrm{Var}\left(\frac{\xi_1 + \xi_2}{\sqrt{2}}\right) = \frac{1}{2}\left(\mathrm{Var}(\xi_1) + \mathrm{Var}(\xi_2)\right) = 1,$$

as we want, since we are assuming that $\mathrm{Var}(\xi_i) = 1$ for all i.

Now the general case should be clear; if N is any positive integer, $\{B_t^N; t \geq 0\}$ will be the random walk that takes a new step every $1/N$ units of time, with step size scaled by the factor $\sqrt{1/N}$:

$$B_t^2 = \begin{cases} 0, & \text{if } 0 \leq t < 1/N; \\ \frac{1}{\sqrt{N}} \sum_{i=1}^{k} \xi_i, & \text{if } k/N \leq t < (k+1)/N. \end{cases}$$

Again, one can check,

$$\mathrm{Var}(B_1^N) = \mathrm{Var}\left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \xi_i\right) = \frac{1}{N}\left(\sum_{i=1}^{N} \mathrm{Var}(\xi_i)\right) = 1.$$

And again, B^N has independent and stationary increments, when restricted to the times $0, 1/N, 2/N, \ldots$, since its steps are independent and identically distributed.

Now imagine that as $N \to \infty$, the random walks B^N have some type of limit B. This can be made precise, but doing so is beyond the scope of these notes. But we can derive the properties this limit should have. We start with the property of normality of increments expressed in requirement (iv) of the definition of Brownian motion. It is a consequence of the Central Limit Theorem. To see this, first fix a $t > 0$. Since the time between steps of the process B^N is $1/N$, the number of steps it takes by time t is $\lfloor tN \rfloor$, which is defined to be the greatest integer less than or equal to tN. That is,

$$B_t^N = \frac{1}{\sqrt{N}} \sum_{i=1}^{\lfloor tN \rfloor} \xi_i.$$

Now, remembering that the mean of each ξ_i is 0 and the variance is 1, the Central Limit Theorem implies that

$$\frac{1}{\sqrt{\lfloor tN \rfloor}} \sum_{i=1}^{\lfloor tN \rfloor} \xi_i \quad \text{converges in distribution to a } N(0,1) \text{ random variable } Z,$$

as $N \to \infty$. Now $B_t^N = \dfrac{\sqrt{\lfloor tN \rfloor}}{\sqrt{N}} \dfrac{1}{\lfloor tN \rfloor} \displaystyle\sum_{i=1}^{\lfloor tN \rfloor} \xi_i$, and $\displaystyle\lim_{N \to \infty} \frac{\sqrt{\lfloor tN \rfloor}}{\sqrt{N}} = \sqrt{t}$. Hence B_t^N converges in distribution to $\sqrt{t}Z$, where $Z \sim N(0,1)$. But $\sqrt{t}Z \sim N(0,t)$, and thus the limit B_t of B_t^N as $N \to \infty$ should be a $N(0,t)$ random variable. This is the motivation for property (iv) in the definition of Brownian motion.

Properties (ii) and (iii) for the limit B follow from the fact that they are true for the approximating random walks B^n. Intuitively, the increment $B_{t+h} - B_t$ is the result of a myriad of small displacements all independent of the history of B_u, $u \leq t$. And the distribution of the total increment $B_{t+h} - B_t$ depends only on the length h of the time interval over which they act.

3. Exercise computing correlations of Brownian motion between different times. The problem is to compute $E[B_s B_t]$ for $0 < s < t$. The idea is to write $B_t = B_t - B_s + B_s$ and to use property (ii) of Brownian motion which implies that B_s and $B_t - B_s$ are independent. We will use conditioning also. To begin

$$E\left[B_s B_t\right] = E\left[B_s(B_t - B_s) + B_s^2\right] = E\Big[E\Big[B_s(B_t - B_s) + B_s^2 \mid B_s\Big]\Big].$$

But

$$\begin{aligned} E\left[B_s(B_t - B_s) + B_s^2 \mid B_s\right] &= B_s E\left[(B_t - B_s)|B_s\right] + B_s^2 \\ &= B_s E\left[(B_t - B_s)\right] + B_s^2 \\ &= B_s^2. \end{aligned} \tag{2}$$

The second equality uses the independence of B_s and $B_t - B_s$, the third uses the fact that the mean of $B_t - B_s = 0$. Substituting this result in (2), and using the fact that $B_s \sim N(0,s)$, gives

$$E\left[B_s B_t\right] = E\left[B_s^2\right] = s.$$

4. Important calculation on exponential moments of a Brownian motion. This calculation further illustrates how to work with Brownian motion. We shall be calculating conditional expectations of the form

$$E\left[X \mid B_u, u \leq s\right]. \tag{3}$$

The conditioning here is with respect to the entire history of the Brownian motion up to time s. Heretofore, we have conditioned only on a finite number of random variable, as in a conditional expectation of the form $E\left[X \mid Y_1, \ldots, Y_n\right]$. We shall not try to define the expectation in (3) rigorously, but will rely on its natural interpretation. The rules we use to handle it will be similar to the rules we used to take expectations conditional on $Y_1, \ldots, Y_n$.

Our object for this exercise is to find, for $s < t$,

$$E\left[e^{\sigma B_t} \mid B_u, u \leq s\right]. \tag{4}$$

The technique will be similar to that used in item 3 above. We will take advantage of the independent increment property and the normality of distributions.

To do this, write $B_t = B_t - B_s + B_s$ and hence $e^{\sigma B_t} = e^{\sigma B_s} e^{\sigma(B_t - B_s)}$. Now, if $\{B_u, u \leq s\}$ is known, $e^{\sigma B_s}$ is certainly known. Moreover by property (ii) and (1), $B_t - B_s$ is independent of $\{B_u, u \leq s\}$ and is a $N(0, t-s)$ random variable. Finally, we know that if $Y \sim N(0, \gamma^2)$, then $E[e^{\lambda Y}] = e^{\lambda^2 \gamma^2 / 2}$. Hence $E\left[e^{\sigma(B_t - B_s)}\right] = e^{(\sigma^2/2)(t-s)}$. Putting all this together:

$$\begin{aligned} E\left[e^{\sigma B_t} \mid B_u, u \leq s\right] &= E\left[e^{\sigma B_s} e^{\sigma(B_t - B_s)} \mid B_u, u \leq s\right] \\ &= e^{\sigma B_s} E\left[e^{\sigma(B_t - B_s)} \mid B_u, u \leq s\right] \\ &= e^{\sigma B_s} E\left[e^{\sigma(B_t - B_s)}\right] \\ &= e^{\sigma B_s} e^{(\sigma^2/2)(t-s)}. \end{aligned} \tag{5}$$

The second equality follows because B_s is known given $\{B_u, u \leq s\}$, the third equality uses the independent increment property, and the final equality was derived already above.

5. The quadratic variation of Brownian motion.

Let $X_t = \sigma B_t$, $t \geq 0$ where B is a standard Brownian motion. Suppose we do not know σ but we can observe X_u over a time interval $0 \leq u \leq t$. We will show that it is possible to determine σ from this observation. In doing so we shall introduce a central concept in the study of Brownian motion, the **quadratic variation**. We shall just state facts here, and dispense with many proofs.

Fix t and let $t_0 < t_1 < t_2 < \cdots < t_n = t$ be a partition of $[0, t]$ into n equal subintervals of size t/n. Let $\delta_i(X) = X_{t_i} - X_{t_{i-1}} = \sigma(B_{t_i} - B_{t_{i-1}}) = \sigma_i(B)$ denote the increment of X on subinterval i of the partition. The **quadratic variation** of X over the partition of $[0, t]$ into n subintervals is defined to be

$$Q_t^n = \sum_{i=1}^{n} (\delta_i(X))^2 = \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}}\right)^2 . \tag{6}$$

We will study this sum as $n \to \infty$. To understand the sum, observe that for each i, $\delta_i(X) \sim N(0, \sigma^2 t_i - t_{i-1}) = N(0, \sigma^2 t/n)$; this is a simple consequence of property (iv) of Brownian motion and the fact $t_i - t_{i-1} = t/n$. Moreover, by the independent increment property of Brownian motion, $\delta_1(X), \ldots, \delta_n(X)$ are independent random variables. As a result, because δ_i has zero mean,

$$E\left[(\delta_i(X))^2\right] = \mathrm{Var}(\delta_i(X)) = \frac{\sigma^2 t}{n}.$$

and

$$E\left[Q_t^n\right] = E\left[\sum_{i=1}^{n} (\delta_i(X))^2\right]] = \sum_{i=1}^{n} \frac{\sigma^2 t}{n} = \sigma^2 t. \tag{7}$$

Think of $\delta_1(X), \ldots, \delta_n(X)$ as n independent observations, in each of which we have an opportunity to learn something about σ^2. It turns out that as $n \to \infty$ we get enough information to learn σ^2 exactly, because Q_t^n converges to $\sigma^2 t$ as $n \to \infty$. The next statement makes this precise in two different senses of convergence.

Theorem 1

(a) $E\left[(Q_t^n - \sigma^2 t)\right] \to 0$ *as* $n \to \infty$.

(b) $\lim_{n \to \infty} Q_t^{2^n} = \sigma^2 t$ *with probability 1.*

The theorem can be extended. Suppose now that $X_s = \nu s + \sigma B_s$, $s \geq 0$, is a Brownian motion with drift ν and variance coefficient σ^2. Define Q_t^n in exactly the same way. It is still true that $\delta_1(X), \ldots, \delta_n(X)$ are independent, but now

$$E\left[Q_t^n\right] = E\left[\sum_{i=1}^{n} (\delta_i(X))^2\right] = \sigma^2 t + \frac{\mu^2 t^2}{n}. \tag{8}$$

(This is left as an exercise.) As $n \to \infty$, the term involving μ disappears and one again recaptures σ^2.

Theorem 2 *Theorem 1 remains true for $X_s = \mu s + B_s$, $s \geq 0$.*

Important remark. Suppose $X_s = \nu s + \sigma B_s$, $s \geq 0$. It can be shown that

$$\lim_{t \to \infty} \frac{X_t}{t} = \nu \quad \text{with probability 1.}$$

This is a large number law generalizing the fact that the average number of heads in n coin flips will tend to the probability of heads as $n \to \infty$. However, it is **not** possible to determine ν with certainty from the observation of X_u on a finite interval $0 \leq u \leq t$.

5. Martingales with respect to Brownian motion.

As short hand for the history $\{B_u, u \leq s\}$ of a Brownian motion up to time s we shall write $\mathcal{F}_s$. Thus

$$E\left[X \mid \mathcal{F}_s\right] \quad \text{means the same as} \quad E\left[X \mid B_u, u \leq s\right].$$

Definition. A random process $\{X_t, t \geq 0\}$ is a martingale relative to $\{\mathcal{F}_t\}_{t \geq 0}$ if:

(a) $E[|X_t|] < \infty$ for all $t \geq 0$;

(b) The value of X_t is known if $\{B_u, u \leq t\}$ is known.

(c) The martingale condition: for any $0 \leq s < t$,

$$E\left[X_t \mid \mathcal{F}_s\right] = X_s \tag{9}$$

Examples. **1.** Brownian motion is itself a martingale, because using independence of increments and the fact that B_s is certainly known if $\{B_u, u \leq s\}$ is

$$E\left[B_t \mid \mathcal{F}_s\right] = E\left[(B_t - B_s) + B_s | \mathcal{F}_s\right] = B_s + B\left[B_t - B_s\right] = B_s.$$

2. If B is a standard Brownian motion, then $X_t = e^{\sigma B_t - \sigma^2 t/2}$ is a martingale. This is an immediate consequence of equality (5) worked out above, for if we multiply this equality by $e^{-\sigma^2 t/2}$ we get

$$E\left[e^{\sigma B_t-\sigma^2 t/2} \mid \mathcal{F}_s\right] =^{\sigma B_s-\sigma^2 s/2} . \tag{10}$$

3. $X_t = B_t^2 - t$ is a martingale. This is an exercise.

Some final remarks on amazing (but true!) facts about Brownian motion.

1. We stated the normality of the increments as part of the definition of Brownian motion. But, a careful and sophisticated use of the Central Limit Theorem show that in fact, property (iv) is a consequence of properties (i), (ii), and (iii) and an assumption that B_1 has mean zero and variance 1. This is neat, because it means that normality of increments is forced by the assumptions of independent stationary increments and path continuity.

2. Let X_t be a random process with continuous paths such that X_t is a martingale and $X_t^2 - t$ is a martingale with respect to conditioning on its past values (see examples 1 and 3 immediately preceding this section.) Then X must be a standard Brownian motion! This Lévy's criterion for Brownian motion.