Financial Mathematics 640:495

Lecture 1 Slides

I. SCOPE OF COURSE

Financial Mathematics, broadly defined, is the application of mathematics to modeling and analysis of financial markets and to aiding management of financial resources.

In this course, We will focus only on analysis of **financial derivatives**, in particular options, for pricing and hedging.

II. FINANCIAL ASSETS AND MARKETS

A. EQUITIES

Basic Equities: Cash, foreign currency, stocks, bonds, money market or bank accounts.

• Basic equities have value in and of themselves.

• They are traded in well-regulated, transparent markets— e.g. stock exchanges.

Basic equities should be compared to **(Physical) Commodities:** These are: **1.** Physical goods, typically raw or partially processed agricultural or mining products, that are commericially traded; **2.** more generally, any economic goods.

II.B. RISK

The primary financial fact of life is **RISK!**

(Loose) Definition. In this course:

risk $\approx$ exposure to **adverse** financial changes

but often, more generally:

risk $\approx$ uncertainty of future prices, values;

Examples:

- stocks are risky assets;
- credit risk: risk that creditors default;
- foreign exchange risk; risk of currency exchange rate fluctuation to cash flows in foreign currency.

Hedging: Acting to reduce or to protect against risk.

II.C. INTEREST

Notation:

- Initial deposit is denoted by V_0.

- Accumulated value at time t by V_t, t measured in years, (assuming no withdrawals).

Compounding.

Formula for the return on interest at nominal annual rate r compounded n times per year:

$$V_t = V_0(1 + \frac{r}{n})^{nt}$$

Formula for the return on continuously compounded interest at nominal annual rate r:

$$V_t = V_0 e^{rt} \quad (= \lim_{n\to\infty} V_0(1 + \frac{r}{n})^{nt}$$

Present value: Let r = interest rate. Let today be $t = 0$. Let $I(t)$ denote 'income' due to us at time t; $I(t) > 0$ means we receive $I(t)$; $I(t) < 0$ means we pay out $-I(t)$. The **present value** of $I(t)$ is its value to us today and this is

$$e^{-rt} I(t).$$

e^{-rt} is called the discount factor.

Generalizing, consider the income stream $I = (I(t_1), I(t_2), \ldots, I(t_n))$ of payments at times $0 \leq t_1 < t_2 < \cdots < t_n$. Its present value is defined to be

$$PV \triangleq \sum_1^n e^{-rt_i} I(t_i).$$

Present value can be used to compare the relative values of different income streams.

II. D. Leverage

Leverage is a financial arrangement which has a multiplying effect on the profit (or loss) on an investment relative to change in value of the invested assets.

We will see examples later.

II.E. FINANCIAL DERIVATIVES

Definition: A **financial derivative** (**derivative security, contingent claim**) is a contract between two parties for a future transaction exchanging cash and/or assets such that:

–the value of the transaction, the gain or loss to either party, depends on the value of other underlying variables at the transaction time.

The underlying variables and how the transactions depend on them are explicitly spelled out in the contract.

Example 1: Forward Contracts

Consider a basic equity or commodity with price, as a function of time, S_t. The equity might be a stock or foreign currency. The commodity might be raw material a firm needs to buy in the future.

In a forward contract on this equity, one party, say A (Alice), agrees to buy the equity from party B (Bob) **at a future time**. No money changes hands at the time A and B enter the contract. The forward contract fixes:

- The amount of asset A will purchase.
- The date T on which the purchase will take place; T is called the **delivery date**.
- The price per unit asset X, called the **delivery price**, of the purchase.

The profit/unit asset realized by A from the contract is called the **payoff** to A. It is

$$\text{payoff/(unit asset) to } A = S_T - X,$$

because A pays X at time T for what is worth S_T.

Example 2: European Call Option on a Stock

Let S_t be the price process of a share of XYZ Corp.

In a European call option on XYZ stock, the **option holder** (Alice) has the right, but not the obligation, to purchase an agreed-upon number of XYZ shares from the option writer (Bob) at a specified time T for a specified price X per share. Alice will pay Bob some **premium** c for this right at the time the contract is entered.

T is the **expiration date**.
X is called the **strike price**.

What will happen? Alice (A) will **exercise the option**—call on Bob to sell at price X—if $S_T > X$. In this case, Alice could turn around and immediately sell for S_T what she bought for X, making a profit of $S_t - X$. But if $S_T \leq X$, Alice can acquire the stock more cheaply in the market and will let the option expire unexercised.

Therefore, the **payoff to A at expiration is:**

$$\text{payoff/share to } A = \max\{S_T - X, 0\}.$$

The **total payoff (net gain) to A is, ignoring discounting:**

$$\max\{S_T - X, 0\} - c$$

where c is the premium that Alice paid for the option. (This is the standard way to write the net payoff, but if one were accounting for discounting and wanted the total net payoff **at** T, one should use instead $\max\{S_T - X, 0\} - ce^{rT}$, where r is the prevailing interest rate for short term loans; but no one seems to use this corrected formula.)

In these examples one sees explicitly how the contract payoff depends upon the unknown future value of an underlying asset. The reason for the name *derivative* or *contingent claim* is precisely that the payoff "derives from" or is "contingent upon" other asset prices.

II.F. SOME TERMINOLOGY

- **Option.** Derivative contract in which one party holds the right to exercise or not.
- **Long position.** The party in the contract that is on the purchasing side of the contracted transaction. In options contracts, the party holding the exercise right has the long position. In forward or futures contracts, the party purchasing and taking delivery is long.
- **Short position.** Counterparty to the long position.
- **Option holder, owner or buyer.** The party in the long position.
- **Option writer.** Party in the short position.
- **Premium.** Price option holder pays to option writer.
- **Expiration date.** Final date at which an option can be exercised.
- **Payoff function.** The profit or loss, as a function of the underlying variables, of the contracted transaction to a party to the contract. In general discussions payoff will mean payoff to the long position.
- **Payoff diagram.** Graph of the payoff function.
- **Equity option.** Option written on an equity (stock) underlying.
- **Index option.** Option written on an index underlying.
- **Open Interest:** The open interest at a particular time for a specified option is the number of listed contracts of that option type held at that time.

II.G. Option types and classification

Options separate broadly into **calls** and **puts**.

- A **call** option gives the holder the option to buy an underlying.
- A **put** option gives the holder the option to sell an underlying.

Options are also classified into two broad types, **vanilla options** and **exotic options**.

• Vanilla options are common, widely available options, often exchange traded, with simple payoff functions based on a single underlying.

• Exotic options are those other than vanilla. They generally depend on underlyings in complicated and, indeed, exotic, ways and are traded over-the-counter.

Some Basic Vanilla Options

• Forward contracts;

• Futures; Futures are essentially forward contracts that are traded. We defer discussion.

• **European calls and puts:** These are call and put options on a single underlying. The designation *European* means the holder may exercise the option only at the expiration date T.

(a) The *European call* at strike X, expiration T, gives the holder the right to **buy** at time T for price X. The payoff to the long position is

$$\max\{S_T - X, 0\}.$$

(b) The *European put* at strike X, expiration T, gives the holder the right to **sell** at time T for price X. The long payoff is

$$\max\{X - S_T, 0\}.$$

• **American calls and puts:** These are again options on a single underlying. The designation *American* means that the holder may exercise the option at any time up to and including the expiration date T. Most exchange traded options are American.

(a) The *American call* at strike X, expiration T, gives the holder the right to **buy** at any time T^*, $T^* \leq T$, for price X. The payoff to the long position is

$$\max\{S_{T*} - X, 0\}.$$

(b) The *American put* at strike X, expiration T, gives the holder the right to **sell** at any time T^*, $T^* \leq T$ for price X. The long payoff is

$$\max\{X - S_{T^*}, 0\}.$$

Note: There is no significance to the appellations *European* or *American.* They could have been called *blue* and *green.*

Exotic options: There are a bewildering variety. Many are **path-dependent**, that is their payoff depends on the whole history of the underlying price up to the time of exercise, not just its price at the time of exercise. We give one example.

Example: **Asian call.** In the Asian call at strike X with expiration T, the holder has the right to ask the option writer for the difference between the average asset price up to time T and the strike. Thus the payoff to the option holder is

$$\max\{\frac{1}{T}\int_0^T S_t\,dt - X, 0\}.$$

II.H. Option markets

• Options are traded on **option exchanges** or **over-the-counter**.

• Option exchanges trade in standardized vanilla options, almost exclusively American calls and puts on stocks and stock indices.
Standardization means that the exchange sets the contract sizes (shares of underlying per option contract), expiration dates, and strike prices. Traders need only negotiate number of contracts and price.
A clearing house acts as the intermediary for all exchange transactions–purchasing, selling, exercising.

• One can close out a position on an option exchange at any time.

Why trade derivatives?

Three reasons are generally given: hedging, speculation, and arbitrage. We will discuss arbitrage later. Investors and financial officers can use derivatives to reduce the risk of transactions they need or want to make in the future. Of course, they pay a price for this either in terms or options premia, or reduced potential gain. Speculators can use derivatives to speculate on future price movements of an asset without investing in that asset itself. Hence derivatives can be used as leverage instruments.

References

-John Hull, **Options, Futures, and Other Derivatives.** Chapter 1

-riskglossary.com: Check out the definitions and explanations of the derivatives and terminology discussed in this lecture at this site.

-888options.com, online tutorial course on basics of options.

-amex.com: This site also has tutorials and a good dictionary.

III. MATH and MODELING BACKGROUND

A. Framework for modeling uncertain markets

Consider a market in M assets A model will do two things, at least:

1. It will specify all possible future histories, that is, outcomes, of the market. Notation:

$$\Omega = \text{the set of market histories.}$$

2. For each asset i, future market outcome ω, and future time t, it will define a price $S_t^{(i)}(\omega)$ for a unit of asset i.

Example: One period, one asset, binomial model. Despite its simplicity, even naiveté, the following model is basic to the course!

• The time periods of the model are $t = 0$ (today, the beginning of the period) and $t = 1$, some unit of time later (the end of the period).

• In the first period, there are two possible market outcomes only, a market upswing, which we denote u, or a market downswing, denoted d.

• If an upswing occurs, the asset return is g.
If a downswing occurs, the asset return is $\ell < g$.

Mathematically this translates to:

$$\begin{aligned}
\Omega &\triangleq \{u, d\} \\
S_0 &\triangleq \text{ today's price, read from market.} \\
S_1(u) &\triangleq gS_0; \\
S_1(d) &\triangleq \ell S_0.
\end{aligned}$$

Example: Extension to two periods:

• Periods $t = 0$, $t = 1$, $t = 2$.
• In each period, an upswing or downswing from previous market state.
• In each period, upswing implies return g, downswing return ℓ.

Model:

$$\begin{aligned}
\Omega &\triangleq \{(u,u),(u,d),(d,u),(d,d)\} \\
S_0 &\triangleq \text{ today's price;} \\
S_1(u,u) &= S_1(u,d) \triangleq gS_0 \\
S_1(d,u) &= S_1(d,d) \triangleq \ell S_0; \\
S_2(u,u) &\triangleq g^2 S_0 \\
S_2(u,d) &= S_2(d,u) \triangleq g\ell S_0 \\
S_2(d,d) &\triangleq \ell^2 S_0
\end{aligned}$$

III.B. Discrete probability spaces

Discrete probability spaces are a framework for modeling an experiment with a random outcome, when the number of possible outcomes is finite. To define a discrete probability space:

1. Define the *outcome space* to be the set of all possible outcomes.

• Notation: $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$.
• Terminology: Subsets of Ω are called *events*. To say, of a trial, "event A occurs" means "the trial's outcome belongs to subset A."

2. To each ω_i in Ω, assign a number p_i, representing the probability that ω_i is the outcome: Require:

(a) for each i, $0 \le p_i \le 1$;

(b) $\displaystyle\sum_1^N p_i = 1$.

For each event A, define

$$I\!P(A) \triangleq \sum_{\omega_i \in \Omega} p_i.$$

More terminology:

- $I\!P$ is called a probability measure on the set of events.
- Ω and $I\!P$ together constitute a *probability space*.

Remark: $I\!P(\{\omega_i\}) = p_i$. We often write this as $I\!P(\omega_i)$.

The entire construction goes through in exactly the same way if Ω is countably infinite: $\Omega = \{\omega_1, \omega_2, \ldots\}$. Requirement (b) becomes

$$\sum_1^\infty p_i = 1.$$

Probability spaces in general. The probability measures, as just defined on finite or countably infinite Ω, satisfy the **finite additivity property:** if $A_1, A_2, \ldots, A_k$ are *disjoint* events,

$$I\!P\left(A_1 \cup \cdots \cup A_k\right) = \sum_{i=1}^k I\!P(A_i). \tag{1}$$

If Ω is countably infinite, $I\!P$ also is *countably additive*: if $A_1, A_2, \ldots,$ are *disjoint* events,

$$I\!P\left(A_1 \cup A_2 \cup \cdots\right) = \sum_{i=1}^\infty I\!P(A_i). \tag{2}$$

In the general definition of a probability space, identities (1) and (2) are taken as *axioms*.

Example. Adding probabilities to the two period, binomial market model.

Recall that in this case $\Omega = \{(u,u), (u,d), (d,u), (d,d)\}$

Model I: After research and observation we think
$I\!P\left((u,u)\right) = \frac{1}{2}, \quad I\!P\left((u,d)\right) = \frac{1}{4}, \quad I\!P\left((d,u)\right) = \frac{1}{8}, \quad I\!P\left((d,d)\right) = \frac{1}{8}.$

Problem. Let A be the event of upswing in the first period. Find $I\!P(A)$.

Note that $A = \{(u,u), (u,d)\}$. Thus $I\!P(A) = I\!P((u,u)) + I\!P((u,d)) = 1/2 + 1/4 = 3/4$.

Model II: (Random Walk, Bull Market) Assume the probability of an upswing in each period is $3/4$ *and* market movements in different periods are *independent*. Then

$I\!P\left((u,u)\right) = (\frac{3}{4})^2, \quad I\!P\left((u,d)\right) = \frac{3}{4}\left(\frac{1}{4}\right),$
$I\!P\left((d,u)\right) = \frac{1}{4}\left(\frac{3}{4}\right), \quad I\!P\left((d,d)\right) = (\frac{1}{4})^2.$

Problem. Find $I\!P$ (at least one upswing).

If B is the event of at least one upswing in the two periods, the complement B^c of B is the event of two downswings, which is the singleton event $\{(d,d)\}$. Thus $I\!P(B) = 1 - I\!P(B^c) = 1 - (1/8) = 7/8$.

III.C. Discrete Random Variables

Object: Model an experiment whose random outcome is a real number in the set $\mathcal{E} = \{y_1, \ldots, y_M\}$.

Approach: Label the outcome of a hypothetical trial by X. X is an example of a **random variable**. The complete description of the behavior of X is given by its **probability mass function**

$$p_X(y), \qquad y \in \mathcal{E},$$

where for each y, $p_X(y)$ gives the probability that X equals y. We write also $I\!P(X\!=\!y)$.

Of course, we require $\sum_{y \in \mathcal{E}} p_X(y) = 1$.

For any subset U of real numbers, we define

$$I\!P\left(X \in U\right) \stackrel{\triangle}{=} \sum_{y \in U} p_X(y).$$

Expectation

The *expected value* or *mean* of X is

$$E\left[X\right] \stackrel{\triangle}{=} \sum_{y \in \mathcal{E}} y p_X(y).$$

The *law of the unconscious statistician* says that for any function g:

$$E\left[g(X)\right] = \sum_{y \in \mathcal{E}} g(y) p_X(y).$$

Example: X is Bernoulli(p) if $I\!P(X\!=\!1) = p$, $\underline{(}X\!=\!1) = p$. Then

$$\mu = E[X] = 0 \cdot (1-p) + 1 \cdot p = b.$$

$$\text{Var}(X) \stackrel{\triangle}{=} E\left[(X-\mu)^2\right] = E[X^2] - p^2 = p(1-p).$$

Functions on probability spaces give r.v.'s

In this course, random variables will often arise as functions defined on a probability space. Here is an example.

Example: This is the random walk, bull market model continued, but now we add prices of a risky asset according to:

$$
\begin{aligned}
\Omega &\triangleq \{(u,u),(u,d),(d,u),(d,d)\} \\
S_1(u,u) &= S_1(u,d) \triangleq gS_0 \\
S_1(d,u) &= S_1(d,d) \triangleq \ell S_0; \\
S_2(u,u) &\triangleq g^2 S_0 \\
S_2(u,d) &= S_2(d,u) \triangleq g\ell S_0 \\
S_2(d,d) &\triangleq \ell^2 S_0 \\
I\!\!P\left((u,u)\right) &= (\frac{3}{4})^2, \quad I\!\!P\left((u,d)\right) = \frac{3}{4}\left(\frac{1}{4}\right) \\
I\!\!P\left((d,u)\right) &= \frac{1}{4}\left(\frac{3}{4}\right), \quad I\!\!P\left((d,d)\right) = (\frac{1}{4})^2.
\end{aligned}
$$

S_1, the price at time 1, and S_2 are random variables! We can compute their probability mass functions from the probability measure $I\!\!P$.

For example, suppose $S_0 = 20$, $g = 1.05$, $\ell = .95$. Then $S_1((u,u)) = S_1(u,d) = 20(1.05) = 21$ and $S_1((d,u)) = S_1((d,d)) = 19$.
The probability mass function of S_1 is

$$p_1(21) = \frac{3}{4} \qquad p_1(19) = \frac{1}{4}.$$

Its expectation is $E[S_1] = 21(3/4) + 19(1/4) = 20.5$.

For S_2: $S_2((u,u)) = 20(1.05)^2 = 22.05$, $S_2((u,d)) = S_2((d,u)) = 20(.95)(1.05) = 19.95$, and $S_2((d,d)) = 20(.95)^2 = 18.05$.
The probability mass function and expectation of S_2 are:

$$p_2(18.05) = \frac{1}{16},\ p_2(19.95) = \frac{6}{16},\ p_3(22.05) = \frac{9}{16}.$$

$$E[S_2] = \frac{18.05}{16} + \frac{6(19.95)}{16} + \frac{9(22.05)}{16} = 21.0125.$$