Financial Mathematics, 640:495: Binomial Trees and Black-Scholes.

1. Purpose.

The purpose of this lecture is to show that the Black-Scholes model may be approximated to arbitrary accuracy by a binomial tree model. There are two reasons to do this. The first is the theoretical one of seeing that there is intellectual coherency between the discrete and continuous time market models and pricing formulae derived so far. We expect the real world to really follow a discrete time model, albeit one with time steps at very close intervals, and the continuous time models we propose should be idealized limits of our discrete time models. The second reason is practical. We know theoretically how to price an option using the Black-Scholes model. If it is a European type option, paying the amount V_T at time T,

$$V_t = \tilde{E}\left[V_T \mid S_u, u \leq t\right],$$

where $\tilde{E}$ means taking expectation assuming that S_t follows a Black-Scholes model with drift r and volatility σ, r being the risk-free interest rate. We saw in the particular case of a call option that we could calculate this conditional expectation explicitly to obtain V_t as a function of S_t. However consider a complicated exotic option, whose payoff depends on the whole path, for example a double barrier knockout that knocks out if the price ever dips below a level K_1 or goes above a level K_2, but otherwise pays $(S_T - X)^+$. Deriving an explicit pricing formula may be very difficult or impossible (actually, I think it can be done in this case but it isn't easy.) If one has an approximate discrete time tree model though, one can run a backward induction algorithm on it to obtain prices. Or consider an American put. There is, as far as I know, no explict formula for its price assuming the Black-Scholes model. But again, one can easily price it using backward induction on an approximate binomial tree model.

This lecture is an attempt to explain, with more details, the material in section 5.7.2 of the Goodman/Stampfli text.

2. A math preliminary. Taylor's formula with remainder (remember that!) applied to Taylor polynomials of order 2, implies

$$e^x = 1 + x + \frac{x^2}{2} + R(x), \quad \text{where } |R(x)| \leq x^3 e^{|x|}/3!. \tag{1}$$

We will use this below to estimate e^x when x is small. For an example that we use later

$$\begin{aligned} e^{\mu\triangle t+\sigma\sqrt{\triangle t}} &= \sigma\sqrt{\triangle t}+(\mu+\frac{\sigma^2}{2})\triangle t \\ &\quad + 2\mu\sigma(\triangle t)^{3/2}+\mu^2(\triangle t)^2+R(\mu\sqrt{\triangle}t+\sigma\sqrt{\triangle t}) \\ &= \sigma\sqrt{\triangle t}+(\mu+\frac{\sigma^2}{2})\triangle t + \bar{R}(\triangle t), \end{aligned} \tag{2}$$

where $\bar{R}(\triangle t)$ denotes the last three terms in the previous expression as a function of $\triangle t$. Note from the estimate on the remainder term in (1) that

$$\lim_{\triangle t \downarrow 0} \frac{\bar{R}(\triangle t)}{(\triangle t)^{3/2}} < \infty.$$

Mathematicians describe this by saying that $\bar{R}(\triangle t)$ is of order $(\triangle t)^{3/2}$ as $\triangle t \to 0$ and by writing (2) as

$$e^{\mu\triangle t+\sigma\sqrt{\triangle t}} = \sigma\sqrt{\triangle t}+(\mu+\frac{\sigma^2}{2})\triangle t + \mathcal{O}((\triangle t)^{3/2}), \tag{3}$$

The student should verify similarly (this is easy) that

$$e^{\mu\triangle t-\sigma\sqrt{\triangle t}} = -\sigma\sqrt{\triangle t}+(\mu+\frac{\sigma^2}{2})\triangle t + \mathcal{O}((\triangle t)^{3/2}), \tag{4}$$

and,

$$e^{\mu\triangle t+\sigma\sqrt{\triangle t}} - e^{\mu\triangle t-\sigma\sqrt{\triangle t}} = 2\sigma\sqrt{\triangle t} + \mathcal{O}((\triangle t)^{3/2}) \tag{5}$$

3. An approximate binomial tree model.

Let assume that a volatility σ and a risk-free interest rate r are given. Our goal is to approximate the Black-Scholes pricing theory with a binomial tree model. Recall that in this theory prices are given as discounted expectations or condition expectations in which it is assumed that the price process $\{S_t\}$ follows a Black-Scholes model with drift r and volatility σ.

Before stating the approximation, note that in the general Black-Scholes model,

$$S_{t+h} = S_t e^{(\mu-\frac{\sigma^2}{2})h+\sigma(B_{t+h}-B_t)} \tag{6}$$

and the normally distributed increment $\sigma(B_{t+h}-B_t)$ in the exponent represents a random fluctuation of variance $E\left[(\sigma(B_{t+h}-B_t))^2\right] = \sigma^2 h$. Intuitively, this represents a random step of absolute size about $\sigma\sqrt{h}$ on average.

We will build the approximate model on a time interval $[0, T]$. Choose a postive integer n and define

$$\triangle t = \frac{T}{n}.$$

This will be the time duration of the periods in the binomial tree price process. The price process will thus be $S_0^n, S_{t_1}^n, \dots, S_{t_n}^n$, where the successive times are

$$t_0 = 0, \quad t_1 = \triangle t, \quad t_2 = 2\triangle t, \dots, \dots, t_n = n\triangle t = T,$$

partitioning $[0, T]$ into n equal subintervals of duration $\triangle t$ each.

The model is specified as follows. The risk-free interest rate is r; σ is a given volatility parameter, and μ' is a given constant. In each period, an upswing means the price changes by the multiplicative factor

$$g = e^{\mu'\triangle t + \sigma\sqrt{\triangle t}};$$

a downswing means it changes by the multiplicative factor

$$\ell = e^{\mu'\triangle t - \sigma\sqrt{\triangle t}}.$$

That is,

$$S_{t_{i+1}}^n = S_{t_i} e^{\mu'\triangle t + \sigma\sqrt{\triangle t}} \quad \text{or} \quad S_{t_{i+1}}^n = S_{t_i} e^{\mu'\triangle t - \sigma\sqrt{\triangle t}}.$$

For motivation, Compare this to (6), but with h replaced by $\triangle t$. The fluctuation about the fixed drift in the exponent in this discrete model has size $\sigma\sqrt{\triangle t}$ to match that in (6).

Recall the risk-neutral measure for this binomial tree model. In each period the probabilities of upswing and downswing are, respectively,

$$\tilde{p}_n = \frac{e^{r\triangle t} - e^{\mu'\triangle t - \sigma\sqrt{\triangle t}}}{e^{\mu'\triangle t + \sigma\sqrt{\triangle t}} - e^{\mu'\triangle t - \sigma\sqrt{\triangle t}}} \quad \text{and} \quad \tilde{q}_n = 1 - \tilde{p}_n \tag{7}$$

Market movements in different time periods are independent.

We will argue that as n gets large, the pricing formulae generated by this model converge to the pricing formulae of the Black-Scholes model with volatility σ. Recall that prices for the binomial tree model are given by expectations and conditional expectations with respect to the risk-neutral measure defined in (7). Prices for the Black-Scholes model are given by taking expectations assuming the price process $\{S_t\}$ is

$$S_t = S_0 \exp\{(r - \sigma^2/2)t + \sigma B_t\}$$

Therefore to show that the binomial tree prices approximate the Black-Scholes prices, it is enough to show that as n gets large, the binomial tree process $\{S^n_{t_i}\}$ under the risk-neutral measure behaves approximately like $\{S_t\}$. To argue this fully is beyond the scope of the course. But we will show that for large n, $S^n_T = S_0 e^{Y_n}$, where Y_n is approximately a normal random variable with mean $(r - \sigma^2/2)t$ and variance $\sigma^2 T$. This is what we want because the exponent in $S_T = S_0 \exp\{(r - \sigma^2/2)T + \sigma B_T\}$ is precisely a normal random variable with mean $(r - \sigma^2/2)t$ and variance $\sigma^2 T$.

Let X_k denote the number of upswings in the first k periods; $k - X_k$ is the corresponding number of downswings. Note that since, under the risk-neutral measure, the probability of upswing in each period is $\tilde{p}_n$ and outcomes in different periods are independent, X_k is a binomial random variable with mean $k\tilde{p}_n$ and variance $k\tilde{p}_n(1-\tilde{p}_n)$. Also, since every upswing contributes a factor of $\sigma\sqrt{\triangle t}$ to the exponent and every downswing a factor of $-\sigma\sqrt{\triangle t}$,

$$\begin{aligned} S^n_{t_k} &= S_0 \exp\{\mu' k \triangle t + X_k \sigma \sqrt{\triangle t} - (k - X_k)\sqrt{\triangle t}\} \\ &= S_0 \exp\{\mu' t_k + 2\sigma\sqrt{\triangle t}(X_k - k/2)\}. \end{aligned} \tag{8}$$

Since we are assuming that markets move up and down independently in different periods, the process $\{S_{t_k}\}$ is a geometric random walk (see the handout at `http://www.math.rutgers.edu/courses/495/495martingales.pdf`). At $T = n\triangle t$, since $\triangle t = T/n$,

$$S^n_T = S_0 \exp\{\mu' T + 2\sigma\sqrt{T/n}(X_n - n/2)\}. \tag{9}$$

The crucial observations are these. First, the Central Limit Theorem applied to binomial random variables implies that for large n,

$$\frac{X_n - n\tilde{p}_n}{\sqrt{n\tilde{p}_n(1-\tilde{p}_n)}} \quad \text{is approximately a } N(0,1) \text{ random variable.} \tag{10}$$

Second, for large n

$$\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\frac{X_n - n\tilde{p}_n}{\sqrt{n\tilde{p}_n(1-\tilde{p}_n)}} \approx \mu' T + 2\sigma\sqrt{T/n}(X_n - n/2) \tag{11}$$

(10) and (11) taken together show that for large n the exponent in the expression for S^n_T in (9) is approximately a normal random variable with mean $(r - (\sigma^2/2))\,T$ and variance $\sigma^2 T$ and this is what we wanted to show.

To finish the demonstration, we need to show (11). This is a bit technical. The idea is to estimate $\tilde{p}_n$ carefully, using the approximations developed in item 2. From item 2 we derive, $e^{t\triangle t} = 1 + r\triangle t + \mathcal{O}(\triangle t^2)$. Also in item 2, we computed an approximation for the denominator of $\tilde{p}_n$ as given in (7). Putting these together,

$$\tilde{p}_n = \frac{1+r\triangle t-(1-\sigma\sqrt{\triangle t}+(\mu'+\frac{\sigma^2}{2})\triangle t) + \mathcal{O}((\triangle t)^{3/2})}{2\sigma\sqrt{\triangle t}+\mathcal{O}((\triangle t)^{3/2})}.$$

Using the tangent line approximation to $1/(1+x)$ at $x = 0$ and Taylor's remainder formula, $1/(1+x) = 1 + \mathcal{O}(x)$ as $x \to 0$, so

$$\frac{1}{2\sigma\sqrt{\triangle t}+\mathcal{O}((\triangle t)^{3/2})} = \frac{1}{2\sigma\sqrt{\triangle t}}\frac{1}{1+\mathcal{O}(\triangle t)} = \frac{1}{2\sigma\sqrt{\triangle t}}\left(1+\mathcal{O}(\triangle t)\right).$$

Substituting this into the expression for $\tilde{p}_n$ and cancelling terms gives

$$\tilde{p}_n = \frac{1}{2} + \sqrt{\triangle t}\frac{r-\frac{\sigma^2}{2}-\mu'}{2\sigma} + \mathcal{O}(\triangle t).$$

From now on we will drop all terms of order $\triangle t$ and write approximations:

$$\begin{aligned}
\tilde{p}_n &\approx \frac{1}{2} + \sqrt{\triangle t}\frac{r-\frac{\sigma^2}{2}-\mu'}{2\sigma} \\
\tilde{p}_n(1-\tilde{p}_n) &\approx \frac{1}{4} - \triangle t\left(\frac{r-\frac{\sigma^2}{2}-\mu'}{2\sigma}\right)^2 \approx \frac{1}{4}
\end{aligned}$$

Hence, using $\triangle t = T/n$,

$$\begin{aligned}
2\sigma\sqrt{T}\frac{X_n - n\tilde{p}_n}{\sqrt{n\tilde{p}_n(1-\tilde{p}_n)}} &\approx 2\sigma\sqrt{T/n}\left(X_n - n\left(\frac{1}{2} + \sqrt{\triangle t}\frac{r-\frac{\sigma^2}{2}-\mu'}{2\sigma}\right)\right) \\
&= 2\sigma\sqrt{T/n}(X_n - n/2) + 2n\sqrt{T/n}\sqrt{T/n}\left(r-\frac{\sigma^2}{2}-\mu'\right) \\
&= 2\sigma\sqrt{T/n}(X_n - n/2)-\left(r-\frac{\sigma^2}{2}\right)T+\mu'T
\end{aligned}$$

Adding $(r - \sigma^2/2)T$ to both sides of the last equation gives (11).

4. Using the binomial tree approximation. You will notice that in the end the parameter μ' in the binomial tree model has no bearing on the limiting distribution of S_t^n, which is that of the Black-Scholes model and depends only on r and σ. Hence we can choose μ' at our convenience. One choice discussed in the text is to take $\mu' = 0$. In this case, the binomial approximation is defined by the parameters n, the factors $g_n = e^{\sigma\sqrt{\triangle t}}$, and $\ell = e^{-\sqrt{\triangle t}}$, with $\triangle t = T/n$. The risk-neutral probability $\tilde{p}_n$, is then, according to the approximation worked out above,

$$\tilde{p}_n = \frac{e^{r\triangle t} - e^{-\sqrt{\triangle t}}}{e^{-\sqrt{\triangle t}} - e^{-\sqrt{\triangle t}}} \approx \frac{1}{2} + \sqrt{\triangle t}\frac{r - \frac{\sigma^2}{2}}{2\sigma}.$$

This is the scaling suggested in 5.7.3 of the text for approximating the Black-Scholes model by a binomial tree.