Financial Mathematics, 640:495: Lecture 2

Sample Problems

1. Alice is short a European put on XYZ stock at strike $30. The selling price is $4. Under what circumstance does Alice gain?

The net payoff of a put to the short position is

$$c - \max\{X - S_T, 0\}$$

where X is the strike, c the premium and S_T the asset price at expiration. For the data of the problem the payoff per stock is thus $4 - \max\{30 - S_T, 0\}$, and this is positive as long as $S_T > \$26$.

2. Bob will receive £1,000,000 in three months. What type of derivative can Bob use to hedge against foreign exchange risk if Bob wants to sell his pounds for dollars when he receives them?

To the class: Following the discussion in class, I changed my mind as to the better solution to this problem, though you could do what I suggested in class. But I think the following is more natural. Let $S_t = $ dollars/pound at time t. Bob is at a disadvantage if S_t declines. He can hedge this by buying a put for selling pounds at some fixed strike price.

Or he could enter the short side of a forward contract on pounds at a fixed exchange rate.

3. In a **range forward** contract with delivery date T and delivery price thresholds $K_1 < K_2$, the holder agrees to buy at price K_1 if $S_T \leq K_1$, to buy at price S_T if $K_1 < S_T < K_2$, and to buy at price K_2 if $S_T > K_2$. Show that the holder's position is equivalent to being short a put at strike K_1 and long a call at strike K_2.

This is easiest to see by drawing the diagram for the payoff at expiration of the range forward and of the portfolio in the options. The payoff at expiration to someone short a put at K_1 and long a call at K_2 is $\max\{S_t - K_2, 0\} - \max\{K_1 - S_T, 0\}$ and a case by case analysis shows

$$\max\{S_t - K_2, 0\} - \max\{K_1 - S_T, 0\} = \begin{cases} S_T - K_1, & \text{if } S_T < K_1; \\ 0, & \text{if } K_1 \leq S_T \leq K_2; \\ S_T - K_2, & \text{if } S_T > K_2. \end{cases}$$

4. A **bull spread** is the position: long one call at strike K_1, short one call at strike K_2, where $K_2 > K_1$. Analyze the payoff at expiration and explain the name.

The payoff function at expiration is

$$\max\{S_T - K_1, 0\} - \max\{S_T - K_2, 0\} = \begin{cases} 0, & \text{if } S_T < K_1; \\ S_T - K_1, & \text{if } K_1 \le S_T \le K_2; \\ K_2 - K_1, & \text{if } S_T > K_2. \end{cases}$$

An investor bullish about the underlying asset might be interested in this spread if he thinks the price will rise above K_1. Of course, such an investor could also just purchase a call at K_1. But if he thinks that it won't rise above level K_2, the bull spread is better because it will cost less, since the investor will spend money to buy the call at K_1, but receive money selling the call at K_2.

Arbitrage and no-arbitrage pricing

Loose Definition of Arbitrage: A trading strategy that earns something from nothing, *no matter how the market evolves in the future.* An arbitrage locks in a riskless profit achieved through trading only.

This definition is loose because we have not been precise about what a "trading strategy" is. We will get to this.

Example: Suppose: • Interest rate $r = 0.05$
• Delivery price on one year forward contracts on fuel oil is \$54/barrel.
• Today's price is \$50/barrel.

Arbitrage strategy: • Borrow \$50 for one year.
• Buy a barrel of oil today.
• Assume a short position in a forward contract.
Value of position today (t=0):

$$Y_0 = 0.$$

Value of position at delivery date T:

$$Y_T = 54 - 50e^{.05} = 54 - 52.56 = 1.44.$$

This is an arbitrage. The riskless profit is \$1.44 per barrel, generated from an initial investment of \$0.

Somewhat more precise definition of arbitrage.

Assume a market model: Ω is the (finite) set of possible future market outcomes.

Given some trading strategy investing in assets of the market, and some initial amount of money Y_0, $Y_t(\omega)$ shall denote the total value of investments by time $t > 0$ when the market outcome is ω;

This strategy is an arbitrage if either:

- $Y_0 < 0$ and for some later time T,
$$Y_T(\omega) \geq 0 \textbf{ for all } \omega \text{ in } \Omega; \text{ or}$$
- $Y_0 = 0$ and for some later time T,
$$Y_T(\omega) \geq 0 \textbf{ for all } \omega \text{ in } \Omega, \text{ and}$$
$$Y_T(\omega) > 0 \text{ for at least one } \omega.$$

We still have to be more precise about a trading strategy, but now we at least have some notation!

The NO ARBITRAGE ASSUMPTION.
Asset prices in a market do not allow arbitrage.

In practice we take this to mean that, over time price movements due to arbitrageurs acting on arbitrage opportunities quickly close arbitrage opportunities. Thus, on average, investors will see prices close to values allowing no arbitrage.

We will now apply this to forward contracts.

Arbitrage and Forward Contracts

Let $r =$ risk-free interest rate. This means one can lend and borrow cash at this interest rate with no risk.

Let $\{S_t\}$ denote the price process of an asset. (Really, it's $\{S_t(\omega)\}$ but ω is not important to the argument.)

We will study a forward contract to buy a unit of asset for price X at time T. The parties enter into the contract at time t.

CLAIM 1. For forward contracts entered into at time t, there is an arbitrage opportunity unless the delivery price X is $F = S_t e^{r(T-t)}$.

Definition $F_t = S_t e^{r(T-t)}$ is called the *forward price.* at time t, interest r, for delivery date T.

Conclusion. In a real forward contract, the delivery price is negotiated so that the contract has zero value at the time it is signed. This means that neither the long or short party feels that it should receive money from the counterparty to enter the contract. The claim implies that this price should be the forward price, because if there is an arbitrage opportunity, the contract will have positive value to one party or the other.

Proof: Let Strategy I be to: (i) go long a forward contract for delivery price X; (ii) borrow a unit of asset, sell it for S_t, and invest it at rate r. Let Y_τ denote the value of this strategy at times $\tau \geq t$.

The cost for the strategy at t is 0, so $Y_t = 0$. At time T, we buy a unit asset for X, return it to the lender; at the same time our cash investment has grown to $S_t e^{r(T-t)}$. Hence, $Y_T = S_t e^{r(T-t)} - X$. This will be an arbitrage unless

$$S_t e^{r(T-t)} - X \leq 0 \quad \text{or} \quad S_t e^{r(T-t)} \leq X. \tag{1}$$

Now consider a second strategy, Strategy II: (i) short a forward contract for delivery price X; (ii) borrow \$ S_t at time t at rate r and use it to purchase a unit of asset for S_t.

Again the initial cost is $Y_t = 0$. But at time T, we receive X and have to return $S_t e^{r(T-r)}$ to the lender, so $Y_T = X - S_t e^{r(T-t)}$. No arbitrage requires

$$X - S_t e^{r(T-t)} \leq 0 \quad \text{or} \quad X \leq S_t e^{r(T-t)} \tag{2}$$

Equations (1) and (2) together imply a no-arbitrage price of $X = S_t e^{r(T-t)}$.

Generalization.

Suppose at time t someone offers you the long position in forward contract on the asset for delivery at time T at price X. Now X is not negotiable and they wish to charge you a C_t for entering the contract ($C_t < 0$ means they pay you to enter the contract). What is C_t? This question is really the same as: what is the value at time t of a forward contract for delivery at time T for price X, X being fixed and not necessarily equal to the forward price?

Again no-arbitrage implies a unique price!

CLAIM 2. At time $t < T$, there is an arbitrage opportunity unless

$$C_t = S_t - X e^{-r(T-t)}.$$

Proof: A. Strategy I: • Sell the contract for C_t.
• Borrow $S_t - C_t$.
• Buy a unit of asset.
    Then $Y_t = 0$.
    But $Y_T = X - (S_t - C_t)e^{r(T-t)}$.
    No arbitrage requires,

$$Y_T = X - (S_t - C_t)e^{r(T-t)} \le 0.$$

or equivalently

$$C_t \le S_t - Xe^{-r(T-t)}.$$

Otherwise put, if $C_t > S_t - Xe^{-r(T-t)}$, then strategy I is an arbitrage.

B. Strategy II: • Borrow a unit of asset and sell it for S_t. • Assume the long position in a forward for C_t. • Invest the remaining $S_t - C_t$ at rate r.
    Note: Strategy II is the exact reverse of Strategy I.
    Then $Y_t = 0$.
But $Y_T = (S_t - C_t)e^{r(T-t)} - X$.
    No arbitrage requires $(S_t - C_t)e^{r(T-t)} - X \le 0$, or equivalently,

$$C_t \ge S_t - Xe^{-r(T-t)}.$$

Otherwise put, if $C_t < S_t - Xe^{-r(T-t)}$, then strategy II is an arbitrage.
    Combining the two inequalities derived, the no-arbitrage price is $C_t = S_t - Xe^{-r(T-t)}$, as claimed.

Using Claim 2, we can re-establish Claim 1, that is, that when the delivery price X is negotiated so that the contract has 0 value to either party at time t, no arbitrage implies X should be set equal to the forward price.
    Indeed, in order for the contract to have value 0, Claim 2 requires

$$0 = S_t - Xe^{-r(T-t)},$$

Hence

$$X = S_t e^{r(T-t)},$$

which is the forward price.

Summary. No arbitrage implies that the delivery price of a forward contract should be the forward price, **under the assumption** (very important!) **that strategies I and II are both allowed.**

This raises the question: what does allowing both strategies I and II entail?

First, that one can borrow and lend at interest rate r.

Second, that one can borrow and sell (short sell) the asset at no additional cost.

The second assumption may not be true if:

• The asset is a dividend-paying stock and the borrower must pay the lender dividends that accrue.

• The asset is a commodity that has storage costs that the borrower must assume.

• The asset is foreign currency that must be repaid at the interest rate for that currency.

NO ARBITRAGE FOR ONE-PERIOD MODELS.

The analysis of forward contracts really required only a one-period model. We were interested in the asset prices only at times t and T, and we made investment decisions only at time t.

We now want to formalize the no-arbitrage argument for one-period models in general. To do this we will use the concept of a portfolio, which we discuss next.

Definition. An investor's portfolio is the list of his or her investments and liabilities and the amount of each.

Assume a market with M investment assets.

Let $\{S_t^{(i)}(\omega)\}$ be the price per unit of asset i, $1 \leq i \leq M$, as a function of time t and market outcome ω.

Mathematically, a portfolio in this market is simply a vector

$$\pi = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_M \end{pmatrix},$$

where π_k is the number of units of asset i in the portfolio.

Note:

• We allow $\pi_k < 0$. This means the investor *owes* $|\pi_k|$ units of asset k to another party.

- The value of portfolio π at time t is

$$\Pi_t(\omega) = \sum_{k=1}^{M} \pi_k S_t^{(k)}(\omega).$$

If we collect the asset prices in a vector

$$S_t(\omega) = \begin{pmatrix} S_t^{(1)}(\omega) \\ S_t^{(2)}(\omega) \\ \vdots \\ S_t^{(M)}(\omega) \end{pmatrix},$$

the value is the inner product

$$\Pi_t(\omega) = \pi \cdot S_t(\omega).$$

In any investment problem we must specify the set of allowable portfolio vectors an investor may choose from.

The set of allowable portfolio vectors is *unconstrained* if π can be *any* M-vector. The assumption of unconstrained portfolio vectors entails the following:

(i) One can purchase or sell assets in any fractional amount.

(ii) One can sell short as much of any asset as one wants.

PORTFOLIOS and ARBITRAGE; one period models.

In this section we will work with a one-period market model that is, a model with two times only, $t = 0$ and $t = \tau > 0$. The set of market outcomes is Ω and the price vectors at times 0 and τ are S_0 and $S_\tau(\omega)$; S_0 is fixed and known; $S_\tau(\omega)$ depends on the market outcome.

In the one period model, the only possible trading strategy is to fix your portfolio at time $t = 0$ and collect on it at time τ. So trading strategies are equivalent to portfolios.

The amount of money invested in the portfolio at $t = 0$ is

$$\Pi_0 = \pi \cdot S_0.$$

The amount of money in the portfolio at time τ is

$$\Pi_\tau(\tau) = \pi \cdot S_t(\omega).$$

What is an arbitrage in this set-up? It is a portfolio that satisfies one of the following two conditions:

(i) $\pi \cdot S_0 < 0$ and $\pi \cdot S_\tau(\omega) \geq 0$ for all ω, or;

(ii) $\pi \cdot S_0 = 0$ and $\pi \cdot S_\tau(\omega) > 0$ for at least one ω, $\pi \cdot S_\tau(\omega) \geq 0$ for all ω.

In the first case, we can erase a debt, or even come out ahead, just by investing in portfolio π.

In the second case, we start with nothing, and at time τ, will net a positive profit for at least one market outcome, and will not go into debt for any market outcome.

Forward Contract Example. As an exercise in the formalism, we derive again the fact that the no-arbitrage delivery price is the forward price.

In this discussion, instead of using the times 0 and τ as the beginning and end of the period, we will use the times t and T. The assets in the market are (a) A riskless bond at interest rate r; (b) A stock; (c) A forward contract in the stock at delivery date T and delivery price X.

$$\bar{S}_t = \begin{pmatrix} 1 \\ S_t \\ 0 \end{pmatrix} \qquad \bar{S}_T(\omega) = \begin{pmatrix} e^{r(T-t)} \\ S_T(\omega) \\ S_T(\omega) - X \end{pmatrix}.$$

We explain $\bar{S}_u$. The first component $\bar{S}_t^1$ is the value at time u of \$1 invested at time t at the risk-free rate r, so $\bar{S}_u^1 = 1$ and $\bar{S}_T^1 = e^{r(T-t)}$. The second component is the price of the underlying asset we are contracting to buy or sell. The last component is the value of the forward contract to the long postion; it is assumed that no money changes hands at time t when the contract is entered, so the value is 0 at time t.

Strategy I of the proof of claim 1 is represented by the portfolio. $\nu^* = (S_t, -1, 1)$; (ν^* is the transpose of ν).
This corresponds to short selling the stock, investing S_t in the bond, owning a forward contract.

Then

$$\nu \cdot S_0 = 0,$$

while
$$\nu \cdot S_\tau(\omega) = S_t e^{r(T-\tau)} - X.$$

No arbitrage requires $S_t e^{r(T-t)} - X \le 0$.

Strategy II of the proof of claim 1 is represented by: $\pi^* = (-S_t, 1, -1) = -\nu^*$. (Borrow S_t, buy a unit of asset, go short one forward contract.)

This time

$\Pi_0 = \pi \cdot S_0 = -S_t \cdot 1 + S_t \cdot 1 = 0$, but $\Pi_\tau(\omega)$ is

$$\begin{aligned}\pi \cdot \bar{S}_\tau(\omega) &= -S_t e^{r(T-\tau)} + S_T(\omega) - (S_T(\omega) - X) \\ &= X - S_t e^{r(T-\tau)}.\end{aligned}$$

No arbitrage implies $X - S_t e^{r(T-\tau)} \le 0$.

Combining this result with that of the previous slide, no arbitrage requires

$$X = S_t e^{r(T-\tau)}.$$

This is the forward price.

AN IMPORTANT PRINCIPLE

We want to establish the following very useful principle:

If two portfolios have the same value at time τ for every market outcome ω, then the assumption of no arbitrage implies they must have the same initial value.

We shall call this the *principle of replicating portfolios* be cause the two portfolios replicate one another in terms of their payoffs at expiration.

Proof idea in words: If two portfolios have the same final value for every market outcome and differ in initial price, we can arbitrage by trading one against the other.

Proof in mathematical symbols.

Let π and ν be two portfolios. To say they have the same value for every market outcome ω is to say

$$\pi \cdot S_\tau(\omega) = \nu \cdot S_t(\omega) \quad \text{for every } \omega. \tag{3}$$

Suppose portfolio π is initially worth strictly less than ν; this means $\pi \cdot S_0 < \mu \cdot S_0$.

Consider the portfolio $\mu = \pi - \nu$. Its initial value is

$$\mu \cdot S_0 = (\pi - \nu) \cdot S_0 = \pi \cdot S_0 - \nu \cdot S_0 < 0, \tag{4}$$

but, from (3),

$$\mu \cdot S_\tau(\omega) = (\pi - \nu) \cdot S_\tau(\omega) = 0 \text{ for every } \omega \tag{5}$$

Hence μ is an arbitrage.

The same argument with the roles of π and μ reversed, likewise shows $\mu \cdot S_0 > \mu \cdot S_0$ enables arbitrage. So, no arbitrage implies $\mu \cdot S_0 = \nu \cdot S_0$.

Forward Contracts Once Again! Return to the problem of the price C_t at time t of a forward contract to buy an asset at time T at price X, where X is given and not necessarily equal to the forward price. Call this contract I.

Suppose one can also enter a forward contract to buy at time T for the forward price, $F = S_t e^{r(T-t)}$. Call this contract II.

Consider the portfolio that at time t is long contract I and short contract II. Its value at t is C_t, since contract II has value 0 at t. At time T this portfolio pays $S_t e^{r(T-t)} - X$, no matter what happens in the market. We could also achieve this payoff by a second portfolio which only invested $e^{-r(T-t)}(S_t e^{r(T-t)} - X) = S_t - X e^{-r(T-t)}$ at time t at the riskless rate r. If there is no arbitrage the two portfolios have the same value at t, so

$$C_t = S_t - X e^{-r(T-t)}.$$

Put-Call Parity for European options

By now you are probably heartily sick of forward contracts, but we will do one more, very important application. Generally this is presented later in such a course but we can do it now and feel very proud of ourselves. It is called put-call parity. It works for European puts and calls (but not American).

Claim 3. Let C_t be the price at time t for a European call with strike X at expiration T. Let P_t be the price of a European put at the same strike and expiration. Then no-arbitrage implies

$$C_t - P_t = S_t - X e^{-r(T-t)}. \tag{6}$$

Notice that we haven't said what P_t and C_t are at this point. But whatever they are, they must satisfy (6), and this identity is put-call parity.

The proof of this claim is as follows. Refer back to problem 4 at the beginning of the lecture. If we apply this problem with $K_1 = K_2 = X$ we see that the payoff at expiration of a forward contract for delivery at price X at time T is the same as the payoff at expiration of the portfolio which is short one European put at strike X and long one European call at strike X. To repeat the argument, the payoff at expiration to the long position of a call is $\max\{S_T - X, 0\}$, the payoff to the short put is $-\max\{X - S_T, 0\}$, so the total payoff is

$$\max\{S_T - X, 0\} - \max\{X - S_T, 0\} = S_T - X,$$

and the last expression is the payoff at expiration to a forward contract with delivery price X. But from Claim 2 we know the value of this forward contract at time t to be $S_t - Xe^{-r(T-t)}$.

On the other hand, the value at time t of being short one put and long one call is $C_t - P_t$, To understand this, suppose you hold a long call; then you could cancel this position by getting a third party to buy this call from you for C_t and take over your long position, because this is what the premium of a call is at t. Similarily, if you wanted a third party to take over your short put position, you would, in effect, have to buy a put from them for P_t. So, all in all, getting out of your position at time t would net you $C_t - P_t$, so this is its value.

Now we apply the replicating portfolio principle. The two portfolios must have the same value at t as they have the same payoff at T for all market outcomes. Thus

$$C_t - P_t = S_t - Xe^{-r(T-t)}.$$

And this finishes the demonstration.