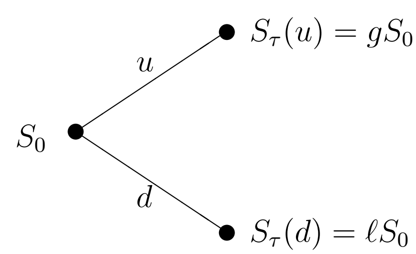
Financial Mathematics, 640:495: Lectures 4 and 5

I. The one-period binomial model.

In these lectures we begin to make a serious study of the one period binomial model with one risky asset and riskless interest rate r. For convenience of terminology, we shall call the risky asset the *stock* and the financial instrument that returns the riskless interest rate the *bond*. Of course, the model applies if the stock is instead a commodity, an index, a futures contract—any risky security—and the bond is instead a bank account or money market.

Let us recall that in the one period, binomial model there are two times $t = 0$ and $t = \tau$ spanning the beginning and end of one period, and two possible market states u and d, which we collect in the outcome space $\Omega = \{d, u\}$. The stock price at time $t = 0$ is denoted S_0, and the possible stock prices at time $t = \tau$ by $S_\tau(u)$ and $S_\tau(d)$, and it is assumed that $S_\tau(u) > S_\tau(d)$. Previously we also introduced the notations $g = S_\tau(u)/S_0$ and $\ell = S_\tau(d)/S_0$ for the returns on the stock for the different outcomes u and d. So we shall sometimes write $S_\tau(d) = \ell S_0$ and $S_\tau(u) = gS_0$, (and sometimes we won't). The interest rate r is the nominal per annum interest rate, so assuming that τ is measured in units of years, the return at time τ on \$1 invested at time $t = 0$, is $e^{r\tau}$.

The picture for our model is



The first result we want to state is very important:

Theorem 1 *The one-period, binomial model with nominal interest rate r, admits no arbitrage if and only if*

$$\ell < e^{r\tau} < g \quad (\textit{equivalently, } S_\tau(d) < S_0 e^{r\tau} < S_\tau(u)). \tag{1}$$

In words, we have no arbitrage if and only if the stock returns more than the riskless interest rate if u occurs and less if d occurs.

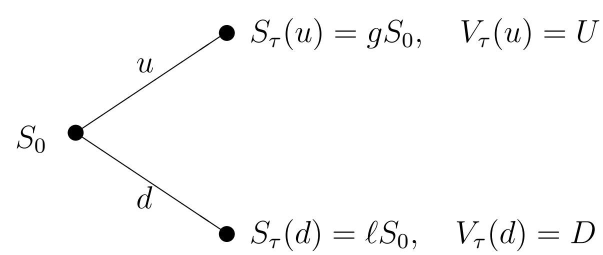
It is not hard to see that if $e^{r\tau} \le \ell < g$ there is an arbitrage to be had by borrowing money at rate r and investing it in the stock. The student should work out carefully an arbitrage portfolio in this case, and in the opposite case $\ell < g \le e^{r\tau}$. (This is a homework problem.) It is somewhat more involved to show that if (1) is true there cannot be an arbitrage portfolio. To do this algebraically is again a homwork problem. In class we present a geometric proof that I am too lazy to incorporate into these notes. So we will say no more about the proof.

From now on we will assume the condition (1), although we will repeat it often to add (I hope) clarity to the discussion.

II. Pricing a general contingent claim

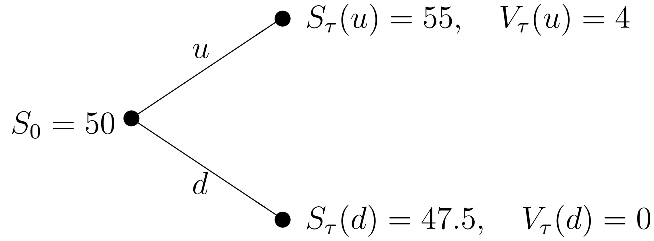
We shall now show how the no-arbitrage principle, in the form of the replicating portfolio principle, tells us how to price a *general* contingent claim in the one-period, binomial model. By a general contingent claim, we just mean a derivative that has a payoff $V_\tau(d) = D$ if outcome d occurs, and a payoff $V_\tau(u) = U$ if outcome u occurs, *where D and U are arbitrary numbers.* We see that no arbitrage implies a unique price for the claim at time $t = 0$ and we will call this price V_0 and give a formula for it.

To summarize our model now, we add V to the previous picture:



Note that it is not assumed here that $U > D$ necessarily.

Here is a concrete example. Consider a stock currently trading at \$50/share that in the next period can rise to \$55 or fall to \$47.50 ($g = 1.1$ and $\ell = 0.95$). Let the contingent claim be a call option with strike \$51. The payoff to the option holder will be $\$55 - 51 = \$4$ per share if u occurs and \$0 if d occurs, so our picture is:



II.A First statement and proof of the pricing formula.

Theorem 2 *Assume that*

$$S_\tau(d) < S_0 e^{r\tau} < S_\tau(u). \tag{2}$$

No-arbitrage implies that

$$V_0 = e^{-r\tau}\left[D\left(\frac{S_\tau(u) - S_0 e^{r\tau}}{S_\tau(u) - S_\tau(d)}\right) + U\left(\frac{S_0 e^{r\tau} - S_\tau(d)}{S_\tau(u) - S_\tau(d)}\right)\right] \tag{3}$$

$$= e^{-r\tau}\left[D\frac{g - e^{r\tau}}{g - \ell} + U\frac{e^{r\tau} - \ell}{g - \ell}\right] \tag{4}$$

$$= \left(S_0 - S_\tau(u)e^{-r\tau}\right)\frac{U - D}{S_\tau(u) - S_\tau(d)} + e^{-r\tau}U. \tag{5}$$

Conversely, if V_0 is defined by one of these formulas, there cannot be arbitrage.

The first two formulas here are just versions of one another. The second formula is derived from the first simply by dividing the numerator and denominator of each fraction by S_0. We write both of them down as the first is more revealing theoretically, but the second is prettier. Your job as a student in this course is to memorize and learn to love (well, at least appreciate) (3). The third formula, formula (5) is the one we will actually derive, and it equals the first two by some algebra, left as an exercise. Formula (5) is called the game theory formula in the text, and it appears there as equation (2.1), although in a slightly different form.

Proof. The text calls this first proof the *game theory* proof. We follow the text closely here so we will be concise. We have tried to keep our notation very close to the text's, but we have not been able to make it exactly the same.

The idea is to construct a portfolio in the stock and the derivative whose payoff Π_τ does not depend on whether u or d happens—that is, whose payoff is riskless. Mathematically this means we require $\Pi_\tau(u) = \Pi_\tau(d)$. Suppose we have a portfolio that achieves this equality, and let Π_τ be the common value of $\Pi_\tau(u)$ and $\Pi_\tau(d)$. We could achieve the same return, by investing $e^{-r\tau}\Pi_\tau$ at the risk free rate. By the replicating portfolio principle, we must have that the value Π_0 of the portfolio at time 0 is

$$\Pi_0 = e^{-r\tau}\Pi_\tau. \tag{6}$$

We shall see that this constraint leads to a unique price for V_0.

To proceed, let b be the number of derivatives and $-a$ the number of stocks we invest in. The value of this portfolio at time 0 is $bV_0 - aS_0$. The value of the portfolio at time τ if u occurs is $bU - aS_\tau(u)$, while if d occurs it is $bD - aS_\tau(d)$. These two must be the same:

$$bU - aS_\tau(u) = bD - aS_\tau(d).$$

This is one linear equation in two unknowns. We can easily show

$$a = \frac{U - D}{S_\tau(u) - S_\tau(d)}, \qquad b = 1,$$

is a solution. Plugging this into (6) gives an equation for V_0 which we can solve to find

$$V_0 = \left(S_0 - S_\tau(u)e^{-r\tau}\right)\frac{U - D}{S_\tau(u) - S_\tau(d)} + e^{-r\tau}U.$$

So far we have not used the condition (2). Of course we need this condition to ensure that there is no arbitrage possible trading the stock against the bond. It is also used to show the converse statement that there is no arbitrage possible using the derivative security if its price V_0 is given by one of the formulas (3)-(5). But we shall delay the proof of this. $\diamond$

The ratio that appears as a in this proof is very important and is called $\triangle$;

$$\triangle \triangleq \frac{U - D}{S_\tau(u) - S_\tau(d)}.$$

(In these notes "$\stackrel{\triangle}{=}$" signifies we are making a definition.) Note that $-\triangle$ is the number of shares of stock to hold in the portfolio of the proof above.

Example. Consider the numerical example of the call option above. Then plugging in to any one of the formulas, say the last one, we find

$$\triangle = \frac{4}{7.5} = \frac{8}{15}, \quad \text{and} \quad V_0 = \$2.54.$$

We want to show explicitly how if the price is less, there is an arbitrage opportunity. The method is one way to do a problem like exercise 4 on page 36 of the text. Suppose you could buy this call option for \$2.00. The option is undervalued, so you should buy it as part of the portfolio in the proof above. This portfolio requires having $-\triangle = -(8/15)$ shares of stock. (We can't really do this, but we could buy 100 calls and have 8 shares of stock and 15 calls for the correct ratio of number of calls to number of stocks.) This means we have short sold 8/15 shares. The proceeds from the short sale are $\$50(\triangle) = \$26 + 2/3$. We use two of these dollars to buy the call and invest the rest, $\$24 + 2/3$ at the risk free rate. Our net worth at time 0 is \$0. Now, if u occurs then at the end of the period we have, from the call a profit of \$4 and from the investment we have $(1.05)(24 + 2/3) = 25.90$, for a total of \$29.90. However we must return 8/15 shares, which are worth $\$55(8/15) = \$29.33$. Our profit is then $\$29.90 - 29.33 = \$0.57$. On the other hand, if d happens, the option expires worthless and we have only the return on the investment \$25.90. But now a share of stock cost \$47.50 to return, we need to return 8/15 shares, so this costs us $\$47.50(8/15) = \$25.33$. We again make the same profit. So we have made an arbitrage. $\diamond$

II.B Second statement and proof of pricing; replication of the derivative and delta hedging.

We will now give a second proof of the formulas in Theorem 2. This time we shall directly prove the formula (3) and elaborate on its significance. We will obtain the formula as follows. We ask for a portfolio that invests only in the stock and the bond and *replicates the payoff of the derivative represented by* V. For completeness, we restate and elaborate in a formal statement.

Theorem 3 *Assume*

$$S_\tau(d) < S_0 e^{r\tau} < S_\tau(u). \tag{7}$$

Let V_τ denote the payoff of any security at time τ, setting $U = V_\tau(u)$, $D = V_\tau(d)$. Then there is a unique portfolio that replicates V_τ. The value of this portfolio at time 0, and hence the value V_0 of the security paying V_τ, is

$$V_0 = e^{-r\tau}\left[D\left(\frac{S_\tau(u) - S_0 e^{r\tau}}{S_\tau(u) - S_\tau(d)}\right) + U\left(\frac{S_0 e^{r\tau} - S_\tau(d)}{S_\tau(u) - S_\tau(d)}\right)\right]. \tag{8}$$

The replicating portfolio invests in $\triangle = (U - D)/(S_\tau(u) - S_\tau(d))$ shares of stock and invests $V_0 - \triangle S_0$ dollars at the riskless rate r. Conversely, if V_0 is given by (8), there cannot be arbitrage.

Proof: We look for a portfolio π that invests in a shares of stock and in b bonds to replicate V_τ. The value of this portfolio if u occurs is $aS_\tau(u) + be^{r\tau}$, which we want to equal U, and if d occurs it is $aS_\tau(d) + be^{r\tau}$, which we want to equal D. Therefore a and b must solve

$$\begin{pmatrix} S_\tau(u) & e^{r\tau} \\ S_\tau(d) & e^{r\tau} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} U \\ D \end{pmatrix}. \tag{9}$$

The condition (7) implies that the determinant of the matrix is non-zero. Matrix inversion and multiplication implies that

$$a = \triangle \qquad \text{and} \qquad b = e^{-r\tau}\left[\frac{S_\tau(u)D - S_\tau(d)U}{S_\tau(u) - S_\tau(d)}\right].$$

The value of this portfolio is $V_0 = aS_0 + b$ and plugging in the values of a and b and some algebra gives (8).

Again, we still have to prove the converse, that if (8) and (7) hold, there cannot be arbitrage, but we delay this. $\diamond$

Theorem 3 is very important. It says in effect that the no-arbitrage price of a derivative is the exact amount of money we need to start with at time 0 in order to replicate the payoff of the derivative trading in the stock and bond, and it tells us how to replicate. Indeed, this motivates the following

Important definition: The strategy of investing in $\triangle$ shares of stock and $V_0 - \triangle(S_0)$ dollars at the risk free interest rate is called the *delta hedge*.

Our example again: We will illustrate Theorem 3 and the use of delta hedging in the numerical example started above. First, as we have observed, for this example,

$$\triangle = \frac{8}{15}.$$

We also compute the factors appearing in (8):

$$\frac{S_\tau(u) - S_0 e^{r\tau}}{S_\tau(u) - S_\tau(d)} = \frac{55 - 52.5}{7.5} = \frac{1}{3} \quad \text{and} \quad \frac{S_0 e^{r\tau} - S_\tau(d)}{S_\tau(u) - S_\tau(d)} = \frac{52.5 - 47.5}{7.5} = \frac{2}{3}.$$

Therefore, because $D = 0$ and $U = 4$, we get

$$V_0 = \frac{1}{1.05}\left[0 \cdot \frac{1}{3} + 4 \cdot \frac{2}{3}\right] = \frac{8}{3.15} = 2.54.$$

Using this example, we can explain the significance of delta hedging. Suppose we have sold 100 contracts on this call option. Each contract is for 100 shares of stock, so we have really sold $10,000$ call options. This puts us in a risky position. If the stock price does rise to \$55, we will have to sell $10,000$ shares for \$51 per share so we will lose \$$40,000$. Delta hedging allows us to completely eliminate the risk of our position. We are paid \$$10,000(V_0) \approx$ \$$25,400$ for the call contracts we sell. Delta hedging tells us that if we purchase $10,000(\triangle)$ shares of stock and invest (or borrow, as the case may be) the remaining \$$10,000V_0 - (10,000)\triangle S_0$ at the risk free rate, our position will earn us \$$40,000$ if the stock price does rise to \$55. Thus we will be able to meet our obligations exactly. Likewise, if the stock price declines to \$47.50, replication means that our portfolio will be worth 0, but this is okay as the options expire worthless and will not require us to pay out.

The numbers in this example work out as follows. The delta hedge tells us to buy $10,000\triangle = 5333$ stocks; this will cost \$$5333(50) =$ \$$266,667$ (rounding to the nearest dollar). To make this purchase we will have to borrow \$$266,667 - 25,400 =$ \$$241,267$, because we have already \$$25,400$ in hand from the sale of the calls. This then is our delta hedge: we own 5,333 shares of stock and have borrowed \$$241,267$ at rate r. $\diamond$

Contemplating the example, you might wonder why anyone would sell the calls. After all, they go through lots of transactions and delta hedging just to come out even. However, the big option markets engage brokers called market makers. Market makers are obliged to quote bid (buy) and ask (sell) prices for all options on the market. This way an investor wanting to enter any option contract will always find a counterparty. In return for this service, the market maker is allowed to maintain a bid-ask spread, that is a difference between the price at which they buy and the price at which they sell to guarantee them a profit. In this idealized model we can imagine that the ask price will be slightly higher than V_0. Returning to the example, suppose we sell the options for \$2.70. We still are at risk and we still use delta hedging. We take \$2.54 from each contract sold and use that in the delta hedge. The hedge exactly replicates the amount of money we will owe the parties that bought the calls. The rest, $\$2.70 - 2.54$ per share, is pure profit that we should invest at the riskless rate. You will get to explore hedging with the assigned exercises from page 38 of the text.

III.C Interpretation of the pricing formula as an expectation: the risk-neutral measure.

Recall from lecture 1 notes, that by assigning probabilities to market outcomes we arrive at probabilistic market models. When we do so, the asset prices and payoffs become random variables. In the one period, binomial models, we need only to assign a probability p to u. Then the probability of d is $1-p$; following standard usage in probability theory, we write $q = 1-p$. Now the asset price S_τ is a random variable taking on value $S_\tau(u)$ with probability p and value $S_\tau(d)$ with probability q. The expectation of S_τ is

$$E[S_\tau] = qS_\tau(d) + pS_\tau(u). \tag{10}$$

Likewise, the payoff V_τ of the general derivative is a random variable taking value U with probability p and value D with probability q. Its expectation is

$$E[V_\tau] = qD + pU. \tag{11}$$

Consider now the formula (8). Ignoring the discount factor, let's denote the factor multiplying U by $\tilde{p}$:

$$\tilde{p} = \frac{S_0 e^{r\tau} - S_\tau(d)}{S_\tau(u) - S_\tau(d)} \tag{12}$$

Let's also denote the factor multiplying D by $\tilde{q}$:

$$\tilde{q} = \frac{S_\tau(u) - S_0 e^{r\tau}}{S_\tau(u) - S_\tau(d)} \tag{13}$$

With these definitions, formula (8) becomes

$$V_0 = e^{-r\tau}\left[D\tilde{q} + U\tilde{p}\right] \tag{14}$$

Now, the assumption that $S_\tau(d) < S_0 e^{r\tau} < S_\tau(u)$ made in (7) implies that both $\tilde{p}$ and $\tilde{q}$ are strictly positive. An easy calculation also shows that

$$\tilde{p} + \tilde{q} = 1.$$

Therefore we can interpret $\tilde{p}$ and $\tilde{q}$ as probabilities! Let us now consider the probability model that assigns probability $\tilde{p}$ to u and probability $\tilde{q}$ to d. We use $\tilde{E}$ to denote expectation using these probabilities. Comparing equation (14) to (11), we see that, *for any derivative*,

$$V_0 = e^{-r\tau}\tilde{E}\left[V_\tau\right] \tag{15}$$

In words, in the one period binomial model *the price at time 0 of* **any** *derivative is the discounted, expected payoff of the derivative assuming* $I\!P(u) = \tilde{p}$ *and* $I\!P(b) = \tilde{q}$.

This is very cool. It applies to any security. For example, consider the asset itself. We can think of this, at least mathematically, as a derivative that pays $U = S_\tau(u)$ if u occurs and $D = S_\tau(d)$ if d occurs. So (15) implies

$$S_0 = e^{-r\tau}\tilde{E}\left[S_\tau\right]. \tag{16}$$

If you didn't quite buy our general reasoning, you can (and should anyway), verify this equation directly using (10) and the definitions of $\tilde{p}$ and $\tilde{q}$. The property expressed in this formula (16) is so fundamental that we make a definition.

Definition. Any probability assignment $I\!P(u) = p$, $I\!P(d) = q$ for which $\tilde{p} > 0$, $\tilde{q} > 0$ and

$$S_0 = e^{-r\tau}E\left[S_\tau\right] \tag{17}$$

is called a *risk-neutral measure* for the one-period, binomial model.

Now here is a simple but neat theorem interpreting our pricing formulas.

Theorem 4 *(i) A risk-neutral measure exists for the one-period, binomial model* if and only if *(our old friend)*

$$S_\tau(d) < S_0 e^{r\tau} < S_\tau(u). \tag{18}$$

When (18) holds the risk-neutral measure is unique and it is given by $\tilde{p}$ and $\tilde{q}$ as defined in (12) and (13).

(ii) Furthermore, when (18) holds, the no-arbitrage price of any derivative with payoff given by the random variable V_τ is

$$V_0 = e^{-r\tau} \tilde{E}\left[V_\tau\right]. \tag{19}$$

Our example yet again. We price the call at strike \$51 in our running example by direct application of the last theorem. According to this result, all we need to do is find the risk-neutral probabilities $\tilde{p}$ and $\tilde{q}$. This requires finding positive $\tilde{p}$ and $\tilde{q}$ that solve

$$\tilde{p} + \tilde{q} = 1, \qquad S_0 = e^{-r\tau}\left[\tilde{q} S_\tau(d) + \tilde{p} S_\tau(u)\right].$$

With our numbers, this is

$$\tilde{p} + \tilde{q} = 1, \qquad 50 = \frac{1}{1.05}\left[\tilde{q}(47.5) + \tilde{p}(55)\right].$$

Since $\tilde{q} = 1 - \tilde{p}$, the second equation is the same as $(1.05)50 = (1-\tilde{p})(47.5) + \tilde{p}(55)$, and it is easy to solve and get $5 = \tilde{p}(7.5)$ or $p = 2/3$. Hence $\tilde{q} = 1/3$. Since $V_\tau(u) = 4$ and $V_\tau(d) = 0$, the pricing formula (19) says

$$V_0 = \frac{1}{1.05}\left[0 \cdot \frac{1}{3} + 4\frac{2}{3}\right] = 2.54. \qquad \diamond$$

Proof of Theorem 4: For $\tilde{p}$ and $\tilde{q}$ to be a risk-neutral measure they must both be positive and must satisfy

$$\begin{aligned} \tilde{p} + \tilde{q} &= 1 \\ S_0 &= e^{-r\tau}\left[\tilde{q} S_\tau(d) + \tilde{p} S_\tau(u)\right] \end{aligned}$$

As long as $S_\tau(d) < S_\tau(u)$, which we are assuming, this set of equations has a unique solution given by

$$\tilde{p} = \frac{S_0 e^{r\tau} - S_\tau(d)}{S_\tau(u) - S_\tau(d)} \quad \text{and} \quad \tilde{q} = \frac{S_\tau(u) - S_0 e^{r\tau}}{S_\tau(u) - S_\tau(d)}.$$

Clearly these are both positive, and hence form a risk-neutral measure, if and only if $S_\tau(d) < S_0 e^{r\tau} < S_\tau(u)$. This proves part (i).

As for part (ii), we have already proved in the previous theorems that formula (19) gives the no-arbitrage price of a derivative. Of course, what we have really only shown so far is that if we assume no-arbitrage, the only possible price of the derivative is given by (19). We have not yet shown the converse, that if this is the price, there is indeed no arbitrage. We can do this now.

Our proof will require a little preparation. Suppose X is a random variable in the one-period binomial model, so X takes two value $X(u)$ or $X(d)$, according as u or d occurs. Suppose $X(u)$ and $X(d)$ are non-negative and at least one of these values is strictly postive. Then $\tilde{E}[X] = \tilde{q}X(d)+\tilde{p}X(u) > 0$, since both $\tilde{p}$ and $\tilde{q}$ are strictly positive. This implies

if $E[X] < 0$, either $X(u) < 0$ or $X(d) < 0$;
if $E[X] = 0$ and $X(u)$ and $X(d)$ are not both 0, either $X(u) < 0$ or $X(d) < 0$.

("Either-or" here means one or the other or both.)

Now suppose that we have a risk-neutral measure, and suppose that we have a portfolio with a shares of stock, \$$b$ invested (or borrowed) at rate r, and c derivatives. Assume V_0 is given by (19). The value of this portfolio at time 0 is $\Pi_0 = aS_0 + b + V_0$. Its value at time τ is the random variable $\Pi_\tau = aS_\tau + be^{r\tau} + V_\tau$. Then from formulas (19) and (17),

$$\begin{aligned} \tilde{E}[\Pi_\tau] &= a\tilde{E}[S_\tau] + be^{r\tau} + cE\left[V_\tau\right] \\ &= aS_0e^{r\tau} + be^{r\tau} + ce^{r\tau}V_0 \\ &= e^{r\tau}\left(aS_0 + b + V_0\right) = e^{r\tau}\Pi_0. \end{aligned}$$

Therefore, if $\Pi_0 < 0$ it follows that $\tilde{E}[\pi_\tau] < 0$, and applying the priniciple discussed above with $X = \Pi_\tau$, we conclude at least one of $\Pi(u)$ or $\Pi_\tau(d)$ is strictly negative. thus there is no arbitrage with any portfolio with $\Pi_0 < 0$.

Similaly, if $\Pi_0 = 0$, it follows that $\tilde{E}\left[\Pi_\tau\right] = 0$. This can only happen if $\Pi_\tau(d) = \Pi_\tau(u) = 0$ or at least one of $\Pi_\tau(d)$ and $\Pi_\tau(u)$ is strictly negative. So we cannot realize a riskless gain starting from zero wealth either. $\diamond$

Final remark. Theorem 4 is a special case for the one-period binomial model of a theorem called the Fundamental Theorem of Asset Pricing. For one period models, not necessarily binomial, this theorem asserts the equivalence of no-arbitrage and the existence of a risk-neutral measure.