640:495 Mathematical Finance; PROBLEMS Fall, 2006

I. Problems on interest and compounding.

Consider an investment of V_0 today that pays back V_1 at the end of one year. We make two definitions. The *total annual return* of this investment is

$$R \triangleq \frac{V_1}{V_0}$$

(Note: the symbol '$\triangleq$' is used in this course when an identity states a definition.) The *annual rate of return* is

$$\bar{r} \triangleq \frac{V_1 - V_0}{V_0}.$$

The following basic identity is easily derived:

$$1 + \bar{r} = R.$$

The *percent annual rate of return* is $100\bar{r}$.

1. (For background, see lecture 1 slides.) Let V_t denote the balance in an interest-bearing account from which no withdrawals are made. Let $\bar{r}$ be the annual rate of return.

(a) Show that if interest in this account is compounded yearly at nominal rate r, then $\bar{r} = r$.

(b) Calculate $\bar{r}$ if instead interest is compounded semi-annually at nominal annual rate r. Calculate $\bar{r}$ if instead interest is compounded continuously at nominal annual rate r.

(c) What would you prefer: an investment at 15 % nominal annual compounded daily, or 15.5% compounded semi-annually?

2. (a) Let $\bar{r}_m$ be the annual rate of return on an investment earning the nominal annual rate r compounded m times yearly? Show that if $m > 1$, then $r_m > r$.

(b) More generally, show that r_m increases as m increases.

3. Present value of a continuous income stream. Let $J(t)$ denote the net amount of cash accumulated between times 0 and t of an income stream. The rate of cash inflow is the derivative $I(t) = J'(t)$. Roughly speaking, this means that the amount of cash received between times t and $t + dt$ is $I'(t)\,dt$. Using Riemann sum approximations of the integral, argue that

$$PV = \int_0^T e^{-rt} I(t)\,\mathrm{d}t.$$

is the appropriate definition of the present value of the income stream.

II. Problem on simple market models. (Refer to lecture 1 slides).

4. (Generalization of the two period market model of Lecture 1, see slides .)

(a) Generalize the two period market model of Lecture 1, slide to three periods. Specifically, assume: (i) in each period the market experiences either an upswing or a downswing from its previous level; (ii) in each period the asset price increases by the factor g in an upswing and by the factor $b < g$ in a downswing. Assume the price/unit of the asset at time 0 is $S_0 = 1$. Write down the outcome space, and, for each ω in the outcome space, and each time t, $t = 1, 2, 3$, the asset price $S_t(\omega)$. How many outcomes (market history paths) are there? How many different prices can the asset have at the final time $t = 3$?

(b) Assume that in each period an upswing occurs with probability 1/3 and a downswing with probability 2/3, and that market movements in different periods are independent. Write down the probabilities of all the different possible market histories (outcomes). Using these probabilities, determine the probability mass function of the random variable S_3 What is its mean?

III. Problems on the payoffs of vanilla options.

5. Suppose that an October European call to buy a share of Apple at \$65 costs \$6.50. If you purchase this, what must happen for you to make a total positive profit (gain from exercise of the option minus premium)? Suppose that instead you buy an October put option at strike \$60 for \$1.50. What must happen in this case for you to exercise? What must the price of Apple be at the October expiry date be for you to make an overall profit?

6. Suppose Bob is long one European put on XYZ stock at strike K_1 and short another put on XYZ at a different strike price K_2, where $K_2 < K_1$, but with the same expiration date. Graph the payoff to Bob at the time of expiration. Bob's position is called a *bear spread.* Explain this terminology, understanding that a *bear market* is a declining market.

7. (For background read the entry on 'leverage' at `riskglossary.com` up to and including the call option example. In this example the call is 'struck at the money'; this means that the strike price of the option is the current market price of the stock.)

(a) Suppose Apple is currently trading at \$65.00. Alice believes Apple will go up in a major way and wants to bet \$6,000 on this speculation. Right now, suppose a January call at strike \$65 costs \$6.00. Therefore Alice can buy 1,000 calls with her money; since Apple options are exchange-traded in standardized contracts for the purchase or sale of 100 shares, this means she can buy 10 contracts. Suppose she buys and holds the option until expiry. Call this investment strategy I. Strategy II is to simply invest the entire \$6,000 today in Apple stock and sell in January at the same date the options expire. (For the sake of simplicity assume that one can purchase fractional amounts of stock.) If the stock price on the date of expiry is \$73 per share, calculate her profit from each different strategy.

Comment: You should find in (a) that the profit for strategy I is significantly greater; this is an example of how an option may provide leverage for speculation. Of course, strategy I also magnifies possible losses. If the option expires worthless, then Alice will lose her entire \$6,000. She would lose everything in strategy II only if the stock price went to 0, which is very unlikely.

(b) For each strategy in (a), determine the profit or loss as a function of the price S_T of Apple stock at the expiration date. Determine that price S_T^* at which both strategies produce the same profit. Alice would have to strongly believe that S_T will be larger than S_T^* to prefer strategy I.

8. A hedger would buy a put on XYZ stock to protect against an increase in XYZ stock price. A speculator would buy a put on XYZ stock to try to profit from a decrease in XYZ stock price. Explain.

IV. Learning about the mechanics and terminology of options markets.

I will not collect this homework. Its purpose is for you to learn how options markets work more concretely. If you know this stuff already, and could anwer the questions below straightaway, you don't need to do this assignment. If you don't, learning this material will help make what we do in class more real! As we proceed in the course, I will assume you understand this material.

Read the first five sections—Introduction, Terminology, Options Revolution, Call/Put Specifics, Premium—of the Options Basics tutorial, available from the web page
`http://www.888options.com/courses/syllabus_options_basics.jsp`;
there is a link to this page from the course resources page
`http://www.math.rutgers.edu/courses/495/resources.html`.
When you get to the page, click on the button "Launch this class in a new window." Answer the following questions.

9. If Alice buys an equity call option from Bob on, say, the American Stock Exchanges Option market, and she chooses to exercise, is it necessarily Bob who must sell her the equity? If not, what happens when Alice goes to exercise?

10. In her brokerage account, Alice is long 10 October call contracts at strike \$60, short 5 October call s at strike \$65 on Apple stock, and short 1 November put on the Nasdaq at strike \$17.50. Which of the following transactions would be closing transactions and which open for Alice?

(i) Entering a long October call contract on Apple at strike \$65.

(ii) Entering a short October call on Apple at strike \$70.

(iii) Entering a short November call on Apple at strike \$60/

(iv) Entering a long November put on Nasdaq at strke \$17.50.

Would any of these transactions completely close out one of Alice's positions in the options market?

11. Go to `http://amex.com` and look at the table of October strikes and calls for Apple. Which are in-the-money and which are out-of-the money? (On the home page menu, click on "Options", and then on the Options page menu, click on "Market Summary". You will get a table of most actively traded options, and Apple is usually on this list. The current price of Apple stock is given in the list. Clicking on the green button in the *option chain* column brings up a table of all the option quotes on Apple stock

More Problems.

12. Suppose you enter a 3 month forward contract on a non-divident paying stock. The price of the stock today is \$ 25 and the risk-free borrowing and lending rate is 6%. What is the forward price?

A month later the price of the stock is \$ 30 and the interest rate is the same. What is the contract worth at this time?

13. In the example on page 6 of the text, calculate the contract price today if the nominal yearly interest rate is 8%, instead of 5.5%.

14. The derivation in class of the forward price of a forward contract assumed that the security provided no income. Suppose instead that the security provides a divident yield at a nominal rate of q per annum. This means that a unit of asset purchased at time t will grow into $e^{q(T-t)}$ units of asset at time $T-t$. If r is the nominal rate of borrowing and lending cash. Show that the forward price at time t for such a security is

$$F_t = S_t e^{(r-q)(T-t)}.$$

15. (Uses problem 14) Suppose the interest rate on British pounds is % 4, while the interest rate on US dollars is % 5. The price today of pounds is \$ 1.9000 and six month forward contracts are available for a delivery price of \$ 1.9300. Is there an arbitrage opportunity and if so, how can one take advantage of it?