Syllabus

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I. **Major Topic: Partial Differential Equations**

1. **First order partial differential equations: Wave-particle duality**

   - Space of 1-jets
   - Standard contact structure and its associated symplectic form in the space of 1-jets
   - Characteristic direction
   - Reduction of a nonlinear first order PDE to a system of nonlinear fisrt order ODE's
   - Local existence

2. **Second order elliptic equations**

   (a) Constant coefficient case: Laplace equation
   - Mean value formula
   - Maximum principle
   - Uniqueness
   - Regularity
   - Harnack inequality
   - Representation formulas

   (b) Sobolev spaces
   - Completeness, reflexivity, density, extensions
   - Gagliardo-Nirenberg-Sobolev inequalities
   - Morrey inequality
   - Poincaré inequality
   - Rellich-Kondrachov compact embedding theorem
   - Border line case: $W_0^{1,n} \hookrightarrow$?

   (c) Weak solutions: L^2 theory
   - Lax-Milgram theorem and Fredholm alternative
   - Existence and uniqueness
   - Regularity

- Maximum principle

3. **Second order elliptic systems: Minimizers of quadratic functionals**

    (a) Convexity
    - Elliptic condition and convexity
    - Legendre-Hadamard condition and quasi-convexity

    (b) Existence
    - Coercivity
    - Weak lower-semicontinuity

    (c) Regularity in 2-d: Morrey theorem
    (d) Regularity for linear elliptic systems
    (e) Regularity for continuous minimizers

II. **Minor Topic: Riemannian Geometry**

1. Differentiable manifolds
    - Tangent and cotangent bundles
    - Vector fields and forms
    - Bracket
    - Immersions and embeddings

2. Riemannian metrics
    - Length
    - Volume

3. Connections
    - Affine connections
    - Riemannian connections
    - Covariant derivative along a curve

4. Geodesics
    - Geodesic equation
    - Geodesic flow
    - Exponential map
    - Gauss lemma
    - Minimizing properties

5. Curvatures
   - Curvature
   - Bianchi identities
   - Sectional curvature
   - Ricci curvature
   - Scalar curvature
6. Jacobi fields
   - Jacobi equation
   - Conjugate points
7. Completeness
   - Hopf-Rinow theorem
   - Hadamard theorem

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