Oral Qualifying Examination for Augusto César Ponce

Topic 1 : Linear Theory for Second Order Elliptic PDE's

The linear theory for second order elliptic PDE's comprises many approaches regarding the study of the problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}, \tag{1}$$

where Ω is a domain in $\mathbb{R}^n$ and the operator L may be in either the nondivergence form

$$Lu := a^{ij}\partial_{ij}u + b^i\partial_i u + c$$

or in the divergence form

$$Lu := \partial_i\big(a^{ij}\partial_j u\big) + b^i\partial_i u + c.$$

In both cases, the functions a^{ij}, defined in Ω, must satisfy the ellipticity condition

$$a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2,\ \forall \xi \in \mathbb{R}^n,$$

for some $\lambda > 0$.

Some questions immediately arise from the study of (2), specially about its well-posedness:

a) what one means by a solution of (2);

b) if there is such a solution;

c) if this solution is unique;

d) if there is a continuous dependence of the solution on the data given in the problem.

In order to answer these questions, a number of results are available to us. The first topic for my Oral Qualifying Examination should include some of them:

- Schauder Theory (classical solutions);
- Sobolev spaces (weak solutions);
- L^p Theory (strong solutions);
- Maximum principles;

- Functional Analysis (Weak topology, Riesz Representation Theorem, Lax-Milgram Theorem and Fredholm Alternative).

References

[1] Brézis, Haïm. *Analyse fonctionnelle* : Théorie et applications. 2nd ed. Paris : Dunod, 1999.

[2] Evans, Lawrence C. *Partial differential equations.* Providence, RI : American Mathematical Society, 1998. (Graduate Studies in Mathematics, 19)

[3] Gilbarg, David; Trudinger, Neil S. *Elliptic Partial Differential Equation of Second Order.* 2nd ed. Berlin : Springer, 1998. (Grundlehren der mathematischen Wissenschaften, 224)

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Topic 2 : The Semilinear Dirichlet Problem

Despite the fact that the linear theory has independent interest, its development is essential if one wants to study more general nonlinear PDE's. As a second topic for my Oral Qualifying Examination, I propose to analyze the semilinear Dirichlet problem

$$\begin{cases} -\triangle u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{2}$$

where Ω is a domain in $\mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}$ is smooth.

I intend to focus on how the linear theory provides us the right abstract space setting in order to guarantee the existence of classical solutions of (2). Some nonlinear methods are readily available in this case:

- Weak convergence techniques;
- Degree theory and fixed point methods;
- Critical Point Theory (minimax characterisation of critical values, in particular the Mountain Pass Theorem);
- Implicit Function Theorem;
- Method of sub and supersolutions.

Under some additional hypotheses on f, we may easily prove the existence of classical solutions of (2) in the following cases (here $0 < \lambda_1 < \lambda_2 \leq \cdots$ denotes the sequence of eigenvalues of $-\triangle$ on Ω, with Dirichlet condition):

- $\limsup_{s \to \pm\infty} \dfrac{f(s)}{s} < \lambda_1$;
- $\lambda_j < \lim_{s \to -\infty} \dfrac{f(s)}{s}, \lim_{s \to +\infty} \dfrac{f(s)}{s} < \lambda_{j+1}$;
- $\lim_{s \to \pm\infty} \dfrac{f(s)}{s} = +\infty$ and $|f(s)| \leq C(|s|^{p-1} + 1)$, for some $2 < p < 2^*$;

- $\lim_{s \to +\infty} \frac{f(s)}{s} = +\infty$, $f > 0$, f is increasing and convex (no bounds on f are imposed); in this case one should replace f by λf, where $\lambda > 0$, and study the set of λ's for which the problem (2) has a nonnegative solution.

References

[1] Brézis, Haïm. *Analyse fonctionnelle* : Théorie et applications. 2nd ed. Paris : Dunod, 1999.

[2] Brézis, Haïm; Nirenberg, Louis. *Nonlinear Functional Analysis and Applications.* (in preparation)

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[4] Kavian, Otared. *Introduction à la théorie des points critiques* et applications aux problèmes elliptiques. Paris : Springer-Verlag, 1993.

[5] Willem, Michel. *Minimax theorems.* Boston : Birkhäuser, 1996.