Proposed topics for the Oral Qualifying Exam
Rutgers University, Department of Mathematics
Candidate: Aaron Lauve
Committee: Earl Taft, Robert Wilson, Vladimir Retakh
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major topic: "Noncommutative Rings"

1 Fundamental Ring Theory

1.1 Semisimplicity

Semisimple rings: definitions and examples; homological characterizations; Wedderburn-Artin theorem (includ. Rieffel's approach); simple rings (artinian and nonartinian). ([Lam91] §§1–3)

J-semisimplicity: characterizations of the Jacobson radical; examples; artinian J-semisimple rings; Hopkins-Levitzki theorem; Nakayama's lemma; von Neuman regular rings as nearly semisimple rings. ([Lam91] §4)

1.2 Skew fields

Basic structure theory: Wedderburn's finite field theorem; commutators as generators; Cartan-Brauer-Hua theorem; "projective lines" and the finite coset question; Herstein's theorem on conjugates. ([Lam91] §13)

Classical constructions: the real quaternions; Hilbert's twisted Laurent series; Dickson's cyclic algebras; the Mal'cev-Neumann construction; $U(\mathfrak{g})$ and it's field of fractions.
([Lam91] §14; [Jac79] ch. 5)

More structure theory: classification of locally compact skew fields; structure of p-fields. ([Wei95] pp. 1–23)

1.3 Free skew fields

As fields of fractions; their universal property; embeddability in larger free skew fields; their center. ([Coh95] pp. 224,225, 235–238)

2 Polynomials

2.1 Symmetric polynomials

The various bases; Jacobi-Trudi identity; Kostka numbers and the Littlewood-Richardson rule; orthonormality of the Schur polynomials. ([FH91] app. A)

2.2 Roots in skew fields

General results: the Euclidean algorithm for skew fields; possibility of infinite number of roots; the Gordon-Motzkin theorem on conjugacy classes.

Minimal polynomials: Dickson's theorem on conjugates; Wedderburn's theorem on complete splitting; criteria for infinite number of roots; the Gelfand-Retakh contribution. ([Lam91] §16; [GR97] §3)

3 Generators and Relations

3.1 Lie algebras

Root systems: reflections in Euclidean space; root systems and simple roots; the Weyl group and Weyl chambers; the Cartan matrix; $\mathfrak{h}^*$. ([Ser01] ch. 5)

Free algebra techniques: Serre's construction of semisimple Lie algebras; the Elimination theorem; the Poincaré-Birkhoff-Witt theorem. ([Hum78] §18; [Reu93] ch. 0; [Jac79] ch. 5)

3.2 Associative algebras

Inversion height in the free skew field; Bergman's diamond lemma. ([Reu96] pp. 93–109; [Ber78] §§1–5)

minor topic: Hopf Algebras

1 Hopf Algebras

Basic theory: coalgebra theory (e.g. fundamental theorem of coalgebras); definition of Hopf algebras and Hopf ideals; the antipode; duality (C^* and A°); density criteria. ([Swe69] §§1.0–4.3, 6.0, 6.1)

Examples: the group algebra and its graded dual; $U(\mathfrak{g})$; the shuffle algebra; Taft algebras. ([DNR01] pp. 158–166; [Swe69] pp. 247–249)

Structure theory: irreducible, simple, and pointed coalgebras; the wedge, coradical, and coradical filtration; Milnor-Moore (the Lie algebra/Hopf algebra correspondence). ([Swe69] §§8.0, 9.0, 13.0)

2 Quantum Groups

Quantum plane: the quantum determinant; $\mathrm{SL}_q(2)$ & $\mathrm{GL}_q(2)$; the quantum plane as comodule. ([Kas95] ch. 4)

Enveloping algebras: definition of $U_q(\mathfrak{g})$; Hopf algebra structure; triangular decomposition; representations (in analogy to those of $U(\mathfrak{g})$; complete reducibility; examples). ([Jan96] ch. 4–5A)

References

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[Coh95] P. M. Cohn. *Skew Fields*. Cambridge University Press, Cambridge, 1995. Theory of General Division Rings.

[DNR01] Sorin Dascalescu, Constantin Nastasescu, and Serban Raianu. *Hopf Algebras*, volume 235 of *Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, Basel, 2001. An Introduction.

[FH91] William Fulton and Joe Harris. *Representation Theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A First Course, Readings in Mathematics.

[GR97] I. Gelfand and V. Retakh. Quasideterminants, I. *Selecta Math. (N.S.)*, 3(4):517–546, 1997.

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[Swe69] Moss E. Sweedler. *Hopf Algebras*. Mathematics Lecture Notes Series. W. A. Benjamin, Inc., New York, 1969.

[Wei95] André Weil. *Basic Number Theory*, volume 144 of *Classics in Mathematics*. Springer-Verlag, Berlin, 1995. Reprint of the second (1973) edition.