Oral Qualifying Exam

Syllabus

Committee: Paul Feehan, Daniel Ocone, Zheng-Chao Han, Richard Gundy

1 Probability

Stochastic processes

Filtrations, conditional expectation, stopping/optional times.

Continuous-time martingales: Doob's Martingale Inequality, Upcrossing Inequalities, convergence theorems (almost sure convergence), uniform integrability, Optional Sampling theorem.

Doob-Meyer decomposition: application to continuous, square-integrable martingales, quadratic variation.

Gaussian processes and Brownian Motion

Levy's construction of Brownian Motion.

(Strong) Markov Property of Brownian Motion: Reflection Principle, distribution of first passage time, distribution of the running maximum of Brownian Motion.

Stochastic Integration

Construction of stochastic integral with respect to continuous square-integrable martingales: approximation by simple processes, extension of stochastic integation with respect to continuous, local martingales.

Change of Variable formula for continuous semimartingales, Martingale Characterization of Brownian Motion, Representation of Brownian Martingales, Girsanov theorem, Novikov's condition.

Stochastic Differential Equations

Strong solution: existence and uniqueness results.

Weak solution: two notions of uniqueness, existence and uniqueness of weak solutions by means of Girsanov's Theorem (a particular case).

Markov Property for time-homogeneous Ito diffusions.

Reference: I. Karatzas, S. Shreve, "Brownian Motion and Stochastic Calculus"

2 Partial Differential Equations

Second Order Elliptic Equations

Harmonic functions: mean-value property, maximum principle, derivative estimates, regularity, analycity, Harnack's inequality.

Poisson Equation: fundamental solution, Green's representation formula, Poincare's method.

Uniformly Elliptic Linear Second Order PDEs: the existence of weak solutions (Lax-Milgram theorem and Fredholm alternative), regularity, maximum principle, Hopf's lemma.

Second Order Parabolic Equations
Heat Equation: fundamental solution, solution in $\mathbb{R}^n$, mean-value theorem, maximum principle, regularity, local estimates of derivatives; solution in bounded domains.
Uniformly Parabolic Linear Second Order PDEs: the existence of weak solutions, Galerkin approximation, regularity.
Reference: L.C. Evans, "Partial Differential Equations"

3 Mathematical Finance

Option pricing in complete markets
Fundamental Theorems of Options Pricing, risk-neutral measure.
Connections with PDEs
Feynman-Kac Theorem, Black-Scholes-Merton Equation.
Options
European Call and Put Option; American Derivative Securities (Perpetual American Put, American Put and Call Option); Exotic Options (Knock-out Barrier, Lookback, and Asian Options).
Volatility models
Heston stochastic volatility model, local volatility, implied volatility.
Jump processes
Poisson process, Compound Poisson process, stochastic integration with respect to jump processes, change of measure.
Reference: S. Shreve, "Stochastic Calculus for Finance"