Topics for oral qualifying exam for Francesco Fiordalisi
Spring, 2011

Major topic: Vertex operator algebras

1. Definitions and properties.
   (a) Formal calculus.
   (b) The notions of vertex algebra and of vertex operator algebra, and basic properties.
   (c) Rationality, commutativity and associativity; equivalence of various formulations, including "weak" formulations.
2. Representations of vertex (operator) algebras.
   (a) The notion of module and basic properties.
   (b) Weak vertex operators.
   (c) The structure of the canonical weak vertex algebra. Local subalgebras and vertex subalgebras of the canonical weak vertex algebra.
   (d) The equivalence between modules and representations.
   (e) General construction theorems for vertex (operator) algebras and modules.
3. Examples of vertex (operator) algebras and modules.
   (a) Vertex (operator) algebras and modules based on the Virasoro algebra.
   (b) Vertex (operator) algebras and modules based on affine Lie algebras.
   (c) Vertex (operator) algebras and modules based on Heisenberg Lie algebras.
   (d) Vertex (operator) algebras and modules on even lattices.
   (e) Vertex operator construction of the affine Lie algebras corresponding to A_n, D_n and E_n.
4. Partition identities, Rogers-Ramanujan recursion.
   (a) Elementary theory of partitions; Euler's partition identity, Jacobi triple product identity, Euler's pentagonal number theorem.

(b) Rogers-Ramanujan identities and Gordon's generalization; Andrews-Gordon identity.

(c) Vertex operator construction of $\widehat{sl_2(\mathbb{C})}$ standard modules, principal subspaces. Level 1 case, Rogers-Ramanujan recursion; higher levels and Rogers-Selberg recursion.

Minor topic: Kac-Moody algebras

1. Poincaré-Birkhoff-Witt theorem.
2. Definitions and properties.
   (a) Root space decompositions.
   (b) The invariant bilinear form and the generalized Casimir element.
   (c) The Weyl group
   (d) Real and imaginary roots, definitions and properties.
3. Affine Lie algebras
   (a) Affine Lie algebras as central extensions of loop algebras
   (b) Classification of affine Lie algebras, twisted and untwisted
   (c) Explicit description of root system and Weyl group
4. Representation theory of Kac-Moody algebras
   (a) Integrable representations of Kac-Moody algebras
   (b) The category O, highest-weight modules and Verma modules
   (c) Formal characters of modules in O
   (d) Integrable highest-weight modules over Kac-Moody algebras, the character formula, the numerator formula and the denominator formula
   (e) Specializations of the character
   (f) Explicit description for affine Lie algebras

References

[A] G. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications, Vol.2, Addison-Wesley, 1976

[CLM] S. Capparelli, J. Lepowsky, and A. Milas, The Rogers-Ramanujan recursion and intertwining operators, Commun. Contemp. Math. 5 (2003)

[CLM2] S. Capparelli, J. Lepowsky, and A. Milas, The Rogers-Selberg recursions, The Gordon-Andrews identities and intertwining operators, The Ramanujan Journal 12 (2006), 379-397.

[FL] A. Feingold, J Lepowsky, The Weyl-Kac Character Formula and Power Series Identities, Adv. in Math. 29 (1978)

[FHL] I. Frenkel, Y.-Z. Huang and J. Lepowsky, On Axiomatic Approaches to Vertex Operator Algebras and Modules, Memoirs Amer. Math. Soc. 104 (1993), 947-966.

[FLM] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, 1988.

[LL] J. Lepowsky and H. Li, Introduction to vertex operator algebras and their representations, Birkhäuser, 2004.

[L] J. Lepowsky, Lectures on Kac-Moody Lie algebras, Université Paris VI, 1978.