Oral Qualifying Exam Syllabus

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I. Combinatorics, Graph Theory, and the Probabilistic Method

1 Combinatorics

Basic Enumeration: counting arguments ([K04, Chapter 1], [vLW, Chapter 13, p. 119ff]),
Stirling's rule ([K04, §2.1], [F, p. 52–54]),
binomial coefficients ([K04, Chapter 1, Chapter 2]),
inclusion-exclusion principle ([K04, Chapter 4], [vLW, Chapter 10, p. 89]),
recurrence relations and generating functions ([K04, Chapter 3], [vLW, Chapter 14, p. 129ff]),

Set Systems: Erdős-Ko-Rado ([vLW, p.56], [Thom, Thm 2.1.9], [Bol, p. 45, §7]),
Sperner's theorem and LYM inequality ([Bol, p. 10], [vLW, p. 54], [Thom, Appendix], [K04, Chapter 5, also p. 124]),
Kruskal-Katona ([Bol, Thm 3, p. 30], [Thom, Apdx. §2.1.5]),
Fisher's and generalized Fisher's inequalities ([Thom, Lem 1.2.1], [vLW, p. 222]),
Ray-Chaudhuri–Wilson and Frankl-Wilson ([Thom, Thm 1.2.3 & 1.2.4], [FW], [vLW, Thm 19.8, p. 222]),
Borsuk Conjecture and Kahn-Kalai counterexample ([Thom, Cor 1.3.2], [KK]),

Lattices and Posets: distributive lattices and geometric lattices ([Bol, p. 145], [K04, Chapter 5, Chapter 9], [Thom, §1.8], [vLW, p. 305ff]),
Birkhoff representation theorem ([AS, bottom of p. 83], [Bol, p. 146]),
Dilworth ([vLW, Thm 6.1], [K04, Thm 5.2.1]),
Möbius inversion ([vLW, Chapter 25]),
Weisner ([vLW, Thm 25.3]),
Dowling-Wilson ([vLW, Thm 25.5]).

Correlation Inequalities: Harris-Kleitman ([Bol, p. 143], [Thom, §1.8, Thm 1.8.1]),
Fortuin-Kasteleyn-Ginibre (FKG inequality) ([Bol, p. 147], [Thom, Thm 1.8.4], [AS, p. 84], [TV, p. 34]),
Ahlswede-Daykin four functions theorem ([Bol, Chapter 19], [Thom, Thm 1.8.2], [AS, p.82]),
application to Shepp's xyz inequality ([S], [AS, §6.4, p. 88]).

Ramsey Theory: Ramsey's theorem ([vLW, p. 28], [Thom, Apdx.], [GRS]),
infinite Ramsey ([Bol, Chapter 20], [GRS]),
probabilistic lower bounds ([AS, p. 16, p. 25, p. 67], [vLW, p. 30]),
Hales-Jewett, van der Waerden, Gallai's theorem ([GRS, Chapter 2], [Gow, Chapter 1]),
statement of Szemerédi's theorem ([TV], [Gow]).

2 Graph Theory

Matching: König ([D, Thm 2.1.1], [K04]),
Hall ([D, Thm 2.1.2], [Bol, p. 5–6], [K04], [vLW, Chapter 5]),
Tutte's 1-factor theorem ([D, Thm 2.2.1], [K04]).

Connectivity: structure of 2-connected graphs ([D, §3.1]),
Menger ([D, §3.3]),
Max-Flow-Min-Cut ([vLW, p. 64], [D, p. 143]),
Kruskal's algorithm (any standard algorithms book).

Planarity: Euler's formula ([D, Thm 4.2.9]),
Kuratowski's theorem ([D, §4.4]),
Wagner's theorem ([D, Thm 4.4.6]).

Coloring: chromatic number and edge chromatic number ([D]),
5 color theorem ([D, Prop 5.1.2]),
Brooks ([vLW, Chapter 3], [D, Thm 5.2.4]),
König edge coloring theorem ([D, p. 119, Prop 5.3.1]),
Vizing's Theorem ([D, Thm 5.3.2]),
perfect graphs ([D]),
Lovász's proof of weak perfect graph conjecture ([D, Thm 5.5.4]).

Extremal Problems: Turán ([D, p. 165], [AS, p. 91]),
statement of regularity lemma ([D, p. 176]),
Erdős-Stone ([D, Thm 7.1.2, cf p. 186–187]).

3 Probabilistic Methods

Basics: probability spaces and random variables ([W]),
linearity of expectation ([AS, Chapter 2.1]),
Bonferroni inequalities ([K04]),
Normal, Binomial, and Poisson distributions ([W], [K05]),
conditional probability and law of total probability ([W]),
Chernoff bound ([AS, Apdx. A], [TV, p. 24, §1.3], [K05]).

Second Moment Method: general procedure ([JLR, §3.1], [AS, Chapter 4], [TV, §1.2]),
Chebyschev's inequality ([AS, p. 41], [TV, p. 19, §1.2]),
application to threshold function for having a certain graph as a subgraph ([JLR, Chapter 3, p. 55ff], [AS, Chapter 4], [K05]).

Lovász Local Lemma: symmetric and general versions ([AS, p. 64–65], [TV, §1.5], [K05]), applications to Latin transversals ([AS, §5.6],[K05]).

Poisson Paradigm: Janson inequalities ([AS, §8.1], [TV, §1.6]),
application to number of triangles in $G_{n,p}$ ([AS, §10.1]),
Brun's sieve ([AS, §8.3], [K05], [JLR, Coro 6.8]),
application to number of isolated vertices in $G_{n,p}$ ([K05], [JLR, Coro 3.31]).

Random Graphs: $G_{n,p}$ versus $G_{n,M}$ ([JLR, p. 2]),
monotone properties ([AS, Chapter 10, §4.4], [JLR, p. 12]),
existence of threshold functions ([JLR, Thm 1.24, §1.5], [AS, p. 156–157]),
connectedness (Erdős-Renyi) ([K05]),
probabilistic refutation of Hajós' conjecture ([K05], [Cat]).

Martingales and Tight Concentration: Azuma's inequality ([AS, §7.2], [JLR, p. 37ff], [K05]),
edge and vertex exposure ([AS, §7.1], [K05], [JLR, p. 39]),
application to concentration of chromatic number ([AS, §7.3]),
Talagrand's inequality ([JLR, p. 39ff], [AS, §7.5,§7.6], [V05]),
comparing Talagrand's and Azuma's inequalities ([JLR, p. 41–43], [AS, §7.7], [V05]),
application to independence number of $G_{n,1/2}$ ([JLR, p. 43]).

II. Additive and Combinatorial Number Theory

Structure of sumsets and applications: basic definitions and results ([V05], [Gow], [TV, p. 6–9,§2.1,§2.2]),
Ruzsa distance and additive energy ([V05], [TV, §2.3]),
theorem showing $|r_1A - r_2A + r_3B - r_4B|$ is "small" when $|A+B|$ is "small" ([V05]),
statement of Balog-Szemerédi-Gowers Theorem ([Gow, §6.4], [TV, §2.5]).

Geometry of Numbers: basic results for a lattices and convex bodies in $\mathbb{R}^d$ ([V05], [Gow], [TV, Chapter 3]),
John's Theorem ([TV, Thm 3.13]),
Ruzsa's Covering Lemma ([TV, Lem 3.14]),
Volume Packing Lemma ([TV, Lem 3.24]),
Blichfeldt's lemma ([Gow], [TV, p. 153]),
Minkowski's first theorem ([V05], [Gow], [TV, p. 153]),
Minkowski's second theorem ([V05], [Gow], [TV, p. 153]),
Discrete John's Theorem ([TV, Lem 3.36]).

Generalized Arithmetic Progressions (GAPs): definitions of proper and non-proper GAPs ([V05], [Gow], [TV, §3.2, and p. 12]),
containing lemmas ([V05], [TV, §3.6, Thm 3.38 & Thm 3.40]).

Freiman's Theorem: Freiman homomorphism ([Gow], [TV2357] (which is [TV, §5.3]), [Gran]),
Freiman's Cube Lemma ([TV2357, p.129, Coro 3.19] (which is [TV, §5.2]), [V06]),
Ruzsa's lemma ([Gow], [TV2357, Lem 3.26]),
Freiman's Theorem ([Gow],[TV2357, Thm 3.32]).

Discrete Fourier Analysis and the Littlewood-Offord problem: Definition $X_{\mathbf{v}}^{(\mu)}$ and basic properties of its Fourier representation ([V05], [TV2357, §4.2, page 168ff] (which is [TV, §7.2])),
Halász-type concentration inequality ([V05], [TV2357, §4.2, page 172ff] (which is [TV, §7.2])),
basic Littlewood-Offord results ([V05], [TV2357, §4.1] (which is [TV, §7.1]), [Bol, p. 19]),
stronger Littlewood-Offord result with Fourier analysis ([V05], [TV2357, §4.2] (which is [TV, §7.2])).

Note: Some of the references are to typed notes ([Gow], [K04], [Thom], and [V05] (partially)), or handwritten notes ([Gran], [V05] (partially), [V06], [K05]). Also note that [TV] and [TV2357] reference *preliminary* drafts of a book, so page and section numbers may differ from the published version (which is due out in 2006).

References

[AS] Alon, Noga; Spencer, Joel H. *The probabilistic method.* Second edition. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience [John Wiley & Sons], New York, 2000. xviii+301 pp.

[Bol] Bollobás, Béla. *Combinatorics: Set systems, hypergraphs, families of vectors and combinatorial probability.* Cambridge University Press, Cambridge, 1986. xii+177 pp.

[Cat] Catlin, Paul A. Hajós' graph-coloring conjecture: variations and counterexamples. *J. Combin. Theory Ser. B* **26** (1979), no. 2, 268–274.

[D] Diestel, Reinhard. *Graph theory.* Third edition. Graduate Texts in Mathematics, 173. Springer-Verlag, Berlin, 2005. xvi+411 pp.

[F] Feller, W. *Introduction to Probability Theory and its Applications, volume 1*, 3rd edition, John Wiley & Sons, Inc., New York, 1968.

[FW] Frankl, P., Wilson, R.M.. Intersection theorems with geometric consequences, *Combinatorica*, v. **1** (1981), p. 357–368.

[Gow] Gowers, Timothy. notes from *Additive and Combinatorial Number Theory*, University of Cambridge, Part III Maths, 2003 Lent Term (notes taken by Philip Matchett Wood).

[GRS] Graham, Ronald L.; Rothschild, Bruce L.; Spencer, Joel H. *Ramsey theory.* Wiley-Interscience Series in Discrete Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1980. ix+174 pp.

[Gran] Granville, Andrew. notes from Workshop on Additive Combinatorics, Centre de recherche mathématiques Université de Montréal, April 2006.

[JLR] Janson, Svante; Łuczak, Tomasz; Rucinski, Andrzej. *Random graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000. xii+333 pp.

[K04] Kahn, Jeff. notes from *Combinatorics I*, Rutgers University, Fall 2004 (notes taken by Philip Matchett Wood).

[K05] Kahn, Jeff. notes from *Probability and Combinatorics*, Rutgers University, Fall 2006 (notes taken by Mike Nieman).

[KK] Kahn, J., Kalai, G.. A counterexample to Borsuk's Conjecture *Bulletin, new series, of the American Mathematical Society*, Volume 29, Number 1, July 1993, Pages 60-62.

[vLW] van Lint, J. H.; Wilson, R. M. *A course in combinatorics.* Second edition. Cambridge University Press, Cambridge, 2001. xiv+602 pp.

[S] Shepp, L. A. The XYZ conjecture and the FKG inequality. *Ann. Probab.* **10** (1982), no. 3, 824–827.

[TV] Tao, Terence, Vu, Van. Additive Combinatorics (Cambridge Studies in Advanced Mathematics). Chapters 1–3 (from sample posted on Tao's website).

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[Thom] Thomason, Andrew. notes from *Extremal Combinatorics*, University of Cambridge, Part III Maths, Lent Term, 2003 (notes taken by Philip Matchett Wood).

[V05] Vu, Van. notes from *Random Structures*, Rutgers University, Spring 2005 (notes taken by Philip Matchett Wood).

[V06] Vu, Van. notes from Workshop on Additive Combinatorics, Centre de recherche mathématiques Université de Montréal, April 2006.

[W] Williams, David. *Probability with martingales.* Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge, 1991. xvi+251 pp.