Oral Exam: Combinatorics, Graph Theory, Probabilistic Methods, Complexity

Cole Franks
Committee: Michael Saks (Chair), Jeff Kahn, Shubhangi Saraf, Swastik Kopparty

April 27, 2015

1 Combinatorics

Enumeration: Bijections, generating functions, binomial and multinomial coefficients, recurrence relations, inclusion-exclusion, Stirling's formula

Hypergraphs: Sperner, LYM inequality, Erdős-Ko-Rado, Kruskal-Katona, Fishers Inequality, Ray-Chaudhuri Wilson, Frankl-Wilson, Baranyai

Posets and Lattices: Dilworth, linear extensions of posets, distributive and geometric lattices, Birkhoff representation theorem

Correlation Inequalities: Harris-Kleitman, Fortuin-Kasteleyn-Ginibre (FKG), Ahlswede-Daykin

Discrepancy: Beck-Fiala, six standard deviations suffice

Ramsey Theory: Ramsey, infinite Ramsey, König tree lemma, probabilistic lower bounds, Van der Waerden, Chvatal-Rödl-Szemerédi-Trotter

Linear Programming: Weak duality theorem, strong duality theorem, fractional coverings and matchings

Algebraic Methods: Combinatorial Nullstellensatz, Schwarz-Zippel Lemma

2 Graph Theory

Matchings: König, Hall, Tutte, stable matchings, matching polytopes

Connectivity: Kruskal's spanning tree algorithm, Menger, max-flow-min-cut, structure of 2-connected graphs

Coloring: 5-color theorem, Brooks, Vizing, Thomassen's 5-list-coloring of planar graphs, perfect graphs, Lovász's proof of weak perfect graph theorem, Galvin's proof of the Dinitz conjecture

Extremal: Turán, statement of the regularity lemma, Erdős-Stone, counting lemma, triangle removal

3 Probabilistic Methods

Basics: Linearity of $\mathbb{E}$, $\cup$-bound and Bonferroni inequalities, Chebyshev's inequality, Chernoff bounds, alteration methods

Second Moment Method: Threshold function for containing a fixed subgraph

Local Lemma: Symmetric and general versions, applications to hypergraph discrepancy, Ramsey lower bounds, Latin transversals, application to SAT

Poisson Paradigm: Jansons inequality, number of triangles in $G_{n,p}$, Bruns sieve, number of isolated vertices in $G_{n,p}$.

Martingales: Vertex and edge exposures, Azuma's inequality and application to chromatic number, Talagrand's inequality

Random Graphs: $G_{n,p}$, $G_{n,M}$, monotone properties, Bollobás-Thomason existence of thresholds

Entropy: Basic properties, Shearer's lemma, Brègman's Theorem

4 Complexity

P vs. NP: Reducibility; the Cook-Levin Theorem; **NP**-completeness of SAT, independent set, 0/1 integer programming, and directed hamiltonian path; conditions that imply $\mathbf{P} \neq \mathbf{NP}$

Diagonalization: Ladner's Theorem, Oracle Turing Machines and the Baker-Gill-Solovay Theorem

Space-bounded complexity: PSPACE completeness of TQBF, **NL** completeness of PATH, Savitch's theorem, the Immerman-Szelepcsényi Theorem

Separation theorems: Deterministic and non-deterministic Time and Space Hierarchy Theorems

Polynomial hierarchy: $\mathbf{\Sigma}_i, \mathbf{\Pi}_i$, complete problems, conditions that lead to the collapse of **PH**.

Circuits: $\mathbf{P} \subset \mathbf{P}_{/\mathbf{poly}}$, Cook-Levin via CKT-SAT, $\mathbf{P}_{/\mathbf{poly}}$ as TMs with advice, Karp-Lipton Theorem, Meyer's Theorem, existence of hard functions, nonuniform hierarchy theorem, $\mathbf{NC}_i, \mathbf{AC}_i$

Randomization: **RP**, **RP** and **ZPP**, error reduction, Sipser-Gacs Theorem, $\mathbf{BPP} \subseteq \mathbf{P}_{/\mathbf{poly}}$, randomized reductions and definition of $\mathbf{BPP} \cdot \mathbf{NP}$

Interactive Proofs: $\mathbf{DIP} = \mathbf{NP}$, GNI $\in \mathbf{AM}$, **NP**-completeness of GI implies $\mathbf{\Sigma}_2^p = \mathbf{\Pi}_2^p$, $\mathbf{IP} = \mathbf{PSPACE}$.

PCP theorem: Equivalence of the three versions, hardness of approximation for MIN-VERTEX-COVER and MAX-INDSET, $\mathbf{NP} \subset \mathbf{PCP}(poly(n), 1)$

Decision Trees: Decision tree complexity, certificate complexity, randomized decision tree complexity, sensitivity, block sensitivity, degree, relationships between $s(f)$, $bs(f)$, $C(f)$, $D(f)$, $deg(f)$, and $R(f)$

Communication Complexity: Fooling sets, tiling lower bound, rank lower bound, discrepancy, $\epsilon(f)$, multiparty communication complexity, GIP_k, n

Lower bounds: Hastad's switching lemma, $\bigoplus \notin \mathbf{AC}\,0$, Razborov-Smolensky theorem, sunflower lemma, monotone-circuit lower bound for CLIQUE