Rutgers University - Graduate Program in Mathematics

Sample Written Qualifying Examination

From deliberations of the Ad Hoc Committee

December 1993 (revised July 1999)

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

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First Day – Part I: Answer each of the following three questions.

Question 1. Let (X_1, d_1) and (X_2, d_2) be metric spaces, and $f : X_1 \to X_2$ a continuous surjective map such that $d_1(p, q) \leq d_2(f(p), f(q))$ for every pair of points $p, q \in X_1$.

(a) If X_1 is complete, must X_2 be complete? Give a proof or a counterexample.

(b) If X_2 is complete, must X_1 be complete? Give a proof or a counterexample.

Question 2. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function, and let $g : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function which is continuous at each x outside a set A of Lebesgue measure zero. Show that $F(t) = \displaystyle\int_{-\infty}^{\infty} f(x)g(xt)dx$ is a continuous function for $t \neq 0$.

Question 3. Let G be a group and H and K subgroups such that H has finite index in G. Prove that $H \cap K$ has finite index in K.

First Day – Part II: Answer <u>three</u> out of the following six questions.

Question 4. Suppose that $f(x)$ is a continuous real-valued function with domain $\mathbb{R}$ which is differentiable for all $x \neq 0$.

(a) If $\lim_{x \to 0} f'(x)$ exists, show that $f'(0)$ exists.

(b) If $\lim_{x \to 0} f'(x)$ need not exist, must $f'(0)$ exist? Prove or give a counterexample.

Question 5. Prove or disprove: there is a real $n \times n$ matrix A such that

$$A^2 + 2A + 5I = 0$$

if and only if n is even. (Here I denotes the $n \times n$ identity matrix).

Question 6. Let f be an analytic function that maps the open unit disk D into itself and vanishes at the origin.

(a) Prove that $|f(z) + f(-z)| \leq 2\,|z^2|$ in D.

(b) Prove that the inequality in 6(a) is strict, except at the origin, unless f has the form $f(z) = \lambda z^2$ for some λ a constant of absolute value one.

Question 7. Let $f(x) = x^5 + 2x^3 + 2x^2 + x - 3$, $g(x) = x^4 + 3x^2 + 2x + 3$. Prove that there is an integer d such that the polynomials $f(x)$ and $g(x)$ have a common root in the field $\mathbb{Q}[\sqrt{d}]$. What is d?

Question 8. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued C^1 functions on $[0, 1]$ such that, for all n,

$$|f_n'(x)| \leq x^{-1/2} \qquad \text{for} \quad (0 < x \leq 1), \text{ and}$$

$$\int_0^1 f_n(x)dx = 0.$$

Prove that the sequence has a subsequence that converges uniformly on $[0, 1]$.

Question 9. Let V be a finite-dimensional linear subspace of $C^{\infty}(\mathbb{R})$ (the space of complex-valued, infinitely differentiable functions). Assume that V is closed under D, the operator of differentiation (i.e., $f \in V \Rightarrow Df = f' \in V$). Prove that there is a constant coefficient differential operator

$$L = \sum_{k=0}^{n} a_k D^k$$

such that V consists of all solutions of the differential equation $Lf = 0$.

Second Day – Part I: Answer each of the following three questions.

Question 1. Evaluate

$$\int_C \frac{e^z}{z(2z+1)^2} dz$$

where C is the unit circle oriented counterclockwise.

Question 2. Let A_n be a sequence of Lebesgue measurable subsets of $[0,1]$, and $A_\infty = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n$ (the set of points that belong to infinitely many of the sets A_n).

(a) Prove that $\sum_{n=1}^{\infty} m(A_n) < \infty$ is a sufficient condition for $m(A_\infty) = 0$.

(b) Prove that $\lim_{n\to\infty} m(A_n) = 0$ is a necessary condition for $m(A_\infty) = 0$.

Which of these two conditions remain valid if we allow A_n to be arbitrary Lebesgue measurable subsets of $\mathbb{R}$?

Question 3. Let A and B be two diagonalizable $n \times n$ complex matrices such that $AB = BA$. Prove that there is a nonsingular $n \times n$ matrix P such that both $P^{-1}AP$ and $P^{-1}BP$ are diagonal matrices.

Second Day – Part II: Answer three of the following six questions.

Question 4. A standard theorem states that a continuous real-valued function on a compact set is bounded. Prove the converse: if K is a subset of $\mathbb{R}^n$, and if every continuous real-valued function on K is bounded, then K is compact.

Question 5. Let p, q, r be continuous real-valued functions on $\mathbb{R}$ with $p(t) > 0$ for all $t \in \mathbb{R}$. Prove that there exist a continuously differentiable function $a(t)$ and a continuous function $b(t)$ such that the differential equation

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0$$

has exactly the same solutions as the equation

$$[a(t)x'(t)]' + b(t)x(t) = 0.$$

Question 6. Let F be a field. Prove that every finite subgroup of the multiplicative group of nonzero elements of F is cyclic.

Question 7. Let I denote the ideal in $\mathbb{Z}[X]$, the ring of polynomials with coefficients in $\mathbb{Z}$, generated by $x^3 + x + 1$ and 5. Is I a prime ideal? Justify your answer.

Question 8. Let O be open and $f : O \to \mathbb{C}$ be holomorphic and one-to-one. Show that for any $z_0 \in O$, the level curves $\Gamma_1 = \{z : Re\, f(z) = Re\, f(z_0)\}$ and $\Gamma_2 = \{z : Im\, f(z) = Im\, f(z_0)\}$ intersect at right angles.

Question 9. Let $\mathbb{R}^2$ represented as 2×1 column vectors be equipped with the Euclidean metric: $d(x, y) = ||x - y||$ where $|| \cdot ||$ is the Euclidean norm. Let T be an isometry (=distance preserving map) of $\mathbb{R}^2$ into itself. Prove that T can be represented as

$$T(x) = a + Ux,$$

where a is a vector in $\mathbb{R}^2$ and U is an orthogonal matrix.