RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2016

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

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Part I. Answer all questions.

1. Let G be an abelian group and for each positive integer n, define

$$G[n] = \{g \in G \mid ng = 0\}.$$

(a) Show that if m and n are positive integers and m divides n, then $G[m] \subseteq G[n]$, and $G[n]/G[m]$ is isomorphic to a subgroup of $G[n/m]$.

(b) Give an example in which m divides n but $G[n]/G[m] \not\cong G[n/m]$. Prove your assertion.

SOLUTION: Obviously $G[n]$ is a subgroup of G. Write $n/m = r \in \mathbb{Z}$. For any $g \in G[m]$, $ng = (rm)g = r(mg) = r0 = 0$, so $g \in G[n]$. Therefore $G[m] \subseteq G[n]$.

Define $f : G[n] \to G$ by $f(g) = mg$. Then f is obviously a homomorphism, and since $r(f(g)) = r(mg) = ng = 0$ for all $g \in G[n]$, the image of f is a subgroup of $G[r]$. The kernel of f is obviously $G[n] \cap G[m] = G[m]$. By the first isomorphism theorem, $G[n]/G[m]$ is isomorphic to the image of f, proving (a).

For (b), let p be a prime and G be cyclic of order p^2. Take $m = p^2$ and $n = p^3$. Then $G[n] = G = G[m]$ so $G[n]/G[m] = 0$. However, $|G[n/m]| = |G[p]| = p$.

2. Let T be a square matrix over $\mathbb{C}$.

(a) Show that if T is invertible and T^k is diagonalizable for some positive integer k, then T is diagonalizable.

(b) Show that the invertibility hypothesis cannot be omitted in (a).

SOLUTION: (a) Replacing T by a similar matrix $U = PTP^{-1}$, we have $U^k = PT^kP^{-1}$ diagonalizable and U invertible, and if U is diagonalizable then so is T. Therefore it suffices to do the case in which T is in Jordan canonical form. Now a matrix in block-diagonal form is diagonalizable if and only if every block is diagonalizable. Therefore we may assume that T is a single Jordan block, of size r, say, and we must prove that $r = 1$. By induction, if $r > 1$, the $(1, 2)$-entry of T^n is $n\lambda^{n-1}$, where λ is the unique eigenvalue of T; also the diagonal entries of T^n are all λ^n. Since T is invertible, $\lambda \neq 0$. Thus $T^k \neq \lambda^k I$. But by assumption PT^kP^{-1} is diagonal for some P, whence $PT^kP^{-1} = \lambda^k I$ and then $T^k = \lambda^k I$, a contradiction. Therefore $r = 1$ and the statement is proved.

ALTERNATIVE SOLUTION:

(a) Since T^k is diagonalizable there exists a monic polynomial $f \in \mathbb{C}[t]$ without multiple roots such that $f(T^k) = 0$. Write $f(t) = \prod_{i=1}^n (t - z_i)$, where $z_1, \dots, z_n$ are distinct. Since T is invertible, so is T^k, and so $z_i \neq 0$ for each $i = 1, \dots, n$. Set $g(t) = f(t^k)$. Then $g(t) = \prod_{i=1}^n (t^k - z_i)$. The roots of $g(t)$ are the kn kth roots of $z_1, \dots, z_n$. Since the z_i's are distinct and none of them equals 0, the kn roots of g are distinct. Finally, $g(T) = f(T^k) = 0$ and as g has no multiple roots, T is diagonalizable.

(b) As a counterexample when the invertibility condition is removed, let $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $T^2 = 0$ is diagonal, but T, which is nonzero with eigenvalues 0 and 0, is not diagonalizable (if it were, it would equal 0).

3. Let I be an ideal in a principal ideal domain R. Show that if $I \neq R$, then

$$\bigcap_{n=1}^{\infty} I^n = (0).$$

(Here I^n is the ideal generated by all products $x_1 \cdots x_n$ such that $x_i \in I$ for all $i = 1, \dots, n$.)

SOLUTION: Assume by way of contradiction that there is $0 \neq x \in \bigcap_{n=1}^{\infty} I^n$. Now $I = Ra$ for some $a \in R$. Then $x \in I^n = Ra^n$ so a^n divides x, for all $n = 1, 2, \dots$. On the other hand R, being a PID, is a UFD, and so x, being nonzero, has only finitely many divisors, up to multiplication by units. Therefore there exist positive integers $m > n$ and a unit u such that $a^m = ua^n$. Hence $a^{m-n} = u$ is a unit. But $a^{m-n} \in I$ so $I = R$, contradiction.

ALTERNATIVE SOLUTION:

As above, proceeding by contradiction, $0 \neq x \in Ra^n$ for all $n = 1, 2, \dots$. So there exist $b_1, b_2, \dots \in R$ such that $x = b_n a^n$ for all n. Then $b_n = b_{n+1}a$ for each n so $Rb_1 \subseteq Rb_2 \subseteq \cdots \subseteq Rb_n \subseteq \cdots$. But R is a PID, hence R is noetherian, so $Rb_n = Rb_{n+1}$ for some n. Then b_n and b_{n+1} divide each other so a must be a unit. But then $I = Ra = R$, contradiction.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let B be a nondegenerate symmetric bilinear form on a 2-dimensional vector space V over the finite field F_p of p elements, where p is prime. Assume that $p \neq 2$. Show that there is always a vector $v \in V$ such that $B(v, v) = 1$.

SOLUTION: Since $p \neq 2$, there is a basis $\{e_1, e_2\}$ of V which is orthogonal ($B(e_1, e_2) = 0$). As B is nondegenerate, $B(e_i, e_i) \neq 0$ for $i = 1$ and $i = 2$. The nonzero squares in F_p^* form a (multiplicative) subgroup of index 2. Fix a non-square $a \in F_p^*$. Replacing the e_i's by appropriate scalar multiples, we may then assume that $B(e_1, e_1)$ is either 1 or a, and the same holds for $B(e_2, e_2)$. If some $B(e_i, e_i) = 1$, we are done, so assume that $B(e_1, e_1) = B(e_2, e_2) = a$.

Then for any scalars x, y, $B(xe_1 + ye_2, xe_1 + ye_2) = (x^2 + y^2)a$ so it is enough to prove that x and y exist in F_p such that $x^2 + y^2 = 1/a$ (and here $1/a$ is, like a, a non-square). If $x^2 + y^2 = c$ is a non-square for some x, y, then $(xz)^2 + (yz)^2 = cz^2 = 1/a$ for suitable $z \in F_p$, as desired. So we may assume that $x^2 + y^2$ is a square for each $x, y \in F_p$. Then the set of all squares (including 0) forms an additive subgroup of F_p of order $1 + ((p-1)/2)$, which contradicts Lagrange's Theorem.

5. Let G be a finite group acting transitively on a set Ω and suppose that $|\Omega| = p^m$ for some prime p and positive integer m. Let P be a Sylow p-subgroup of G (for the same prime p). Prove: P acts transitively on Ω.

SOLUTION:

Write $|G| = p^n a$, with $(p, a) = 1$. Thus, $|P| = p^n$. Fix $\alpha \in \Omega$ and set $H = G_\alpha$. Then $|G : H| = |\Omega| = p^m$ so $|H| = p^{n-m}a$. Then $P_\alpha = P \cap H$ has order p^c with $c \leq n - m$, by Lagrange. So $|P : P_\alpha| = p^n/p^c \geq p^m = |\Omega|$. As P acts on Ω, the P-orbit containing α therefore has cardinality at least $|\Omega|$, so that P-orbit is all of Ω. Therefore P acts transitively on Ω.

ALTERNATIVE SOLUTION:

Again fix $\alpha \in \Omega$ and set $H = G_\alpha$. Consider $PH = \{xh \mid x \in P,\ h \in H\}$. ($PH$ need not be a subgroup of G, a priori.) Then PH is a union of right cosets of P so $|P|$ divides $|PH|$. Likewise PH is a union of left cosets of H so $|H|$ divides $|PH|$. Therefore $|PH|$ is divisible by l.c.m.$(|P|, |H|)$. But $|G|/|H| = |G : H| = |\Omega|$ is a power of p by assumption, and $|G|/|H|$ divides $|G|$, so $|G|/|H|$ divides $|P|$. Therefore l.c.m.$(|P|, |H|) = |G|$, so $PH = G$.

Now choose any $\beta \in \Omega$. Since G is transitive on Ω, there is $g \in G$ such that $g.\alpha = \beta$. Write $g = xh$ with $x \in P$ and $h \in H$. Then $\beta = xh.\alpha = x.(h.\alpha) = x.\alpha$. As $\beta \in \Omega$ was arbitrary and $x \in P$, P is transitive on Ω.

End of Session 1

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Written Qualifying Examination

August 2016

Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

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Part I. Answer all questions.

1. Use a contour integral to evaluate

$$\int_0^{2\pi} \frac{d\theta}{(2+\cos(\theta))^2}.$$

SOLUTION: Set $z = e^{i\theta}$*, so that* $(z+z^{-1})/2 = \cos\theta$*. Then* $dz/(iz) = d\theta$*, and*

$$I = \int_0^{2\pi} \frac{d\theta}{(2+\cos(\theta))^2} = \int_C \frac{dz}{iz(2+(z+z^{-1})/2)^2},$$

where C *is the positively-oriented unit circle. Simplification gives*

$$I = \frac{4}{i}\int_C \frac{z dz}{(z^2+4z+1)^2}.$$

The discriminant of the quadratic is $16-4>0$*, so the roots are real; they in fact are:*

$$\frac{-4\pm\sqrt{12}}{2} = -2\pm\sqrt{3} = \alpha_{\pm},$$

say. The root α_+ *is in the unit circle, so contributes by the residue theorem*

$$I = \frac{4}{i}\, 2\pi i\ \mathrm{Res}_{z=\alpha_+}\ \frac{z}{(z-\alpha_+)^2(z-\alpha_-)^2} = 8\pi\ \frac{d}{dz}\frac{z}{(z-\alpha_-)^2}\bigg|_{z=\alpha_+}$$

$$= 8\pi\ \frac{(z-\alpha_-)^2 - 2z(z-\alpha_-)}{(z-\alpha_-)^4}\bigg|_{z=\alpha_+} = 8\pi\frac{-(\alpha_+ + \alpha_-)}{(\alpha_+ - \alpha_-)^3}$$

$$= 8\pi\frac{4}{(2\sqrt{3})^3} = \frac{4\pi}{3^{3/2}}.$$

2. Write $z = x+iy$ and let $R = \{(x,y):\ (x-1)^2+y^2<1, y>x\}$. Find a biholomorphic map from R to the unit disk $D = \{(x,y):\ x^2+y^2<1\}$. You may express your answer as a composition of explicitly given biholomorphic maps.

SOLUTION: We first notice that the line $y=x$ *intersects the circle* $(x-1)^2+y^2=1$ *at* $z=0$ *and* $z=1+i$*. Also notice that a linear fractional map* *sends a generalized circle (including a straight line) to a generalized circle. Now define*

$$w = \phi_1(z) = \frac{z}{1 - \frac{z}{1+i}}.$$

Since $\phi_1(0) = 0$, *and* $\phi_1(1+i) = \infty$, ϕ_1 *maps the boundary of* R *to two rays originating from* 0. *Noting* $\phi_1(\frac{1+i}{2}) = 1 + i$, ϕ_1 *maps the straight edge of* R *to the ray* $\{z = re^{i\frac{\pi}{4}} : r > 0\}$ *and maps* R *into sector* $R_1 = \{z = re^{i\theta} : r > 0,\ \pi/4 < \theta < \pi/2\}$*(as the* $\frac{\pi}{4}$ *angle between the straight edge of* R *and its circular arc is preserved by* ϕ_1*). Now, define*

$$w = \phi_2(z) = e^{-i\pi/4}z.$$

Then ϕ_2 *maps* R_1 *to* $R_2 = \{z = re^{i\theta} : r > 0,\ 0 < \theta < \pi/4\}$. *Define* $\phi_3(z) = z^4$. *Then* ϕ_3 *maps* R_2 *to the upper-half plane* $R_4 = \{z = re^{i\theta} : r > 0,\ 0 < \theta < \pi\}$. *Finally* $\phi_4 = \frac{z-i}{z+i}$ *maps* R_4 *to the unit disk* D. *Hence* $F = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$ *biholomorphically maps* R *to the unit disk* D.

3. Let $u(x, y)$ be a harmonic function on the unit disk $D = \{z : |z| < 1|\}$; specifically, we assume that u is twice continuously differentiable on D and that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where $z = x + iy$.

(a) Show that $f(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ is a holomorphic function on D.

(b) For any piecewise smooth curve $\gamma \subset D$ connecting 0 to $z \in D$, define $F(z) = \int_\gamma f(z)dz$. Prove that F is a well defined holomorphic function on D.

(c) Show that $\mathrm{Re}(F(z)) = u(z) - u(0)$.

SOLUTION: (a) Because the real and imaginary parts of $f(z)$ *have continuous first partial derivatives, we may show that* $f(z)$ *is a holomorphic function by verifying that* $f(z)$ *satisfies the Cauchy-Riemann equations: as* u *is harmonic, we have*

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right); \quad \textit{also} \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right).$$

(b) Since f is holomorphic and D is simply connected, Cauchy's theorem applied in D implies that the line integral is independent of the choice of the path so that $F(z)$ is well defined.

(There is also a direct argument for Cauchy's theorem in a disc without having to deal with the topological notion of simply connectedness: using line integrals along radii in D and Goursat's theorem on triangles in D to prove the existence of a holomorphic primitive $G(z)$ of $f(z)$ in D, namely, $G'(z) = f(z)$ for $z \in D$. Then for any piecewise smooth curve $\gamma \subset D$ connecting 0 to $z \in D$, $G(\gamma(t))$ is piecewise continuously differentiable in t, with $\frac{d}{dt}G(\gamma(t)) = G'(\gamma(t))\gamma'(t)$, so that $\int_\gamma f(z)dz = \int G'(\gamma(t))\gamma'(t)dt = \int \frac{d}{dt}G(\gamma(t))dt = G(z) - G(0)$, which shows that $\int_\gamma f(z)dz$ is independent of the choice of path from 0 to z. Here is yet another, more direct approach: since f is given in terms of u, we express $\int_\gamma f(z)dz$ in terms of u as follows.

$$\begin{aligned}\int_\gamma f(z)dz &= \int_\gamma (u_x - iu_y)(dx + idy) \\ &= \int_\gamma \big\{\big(u_x dx + u_y dy\big) + i\big(-u_y dx + u_x dy\big)\big\}\end{aligned}$$

where $\int_\gamma u_x dx + u_y dy = \int_\gamma \nabla u \cdot \vec{d\gamma} = u(\gamma(1)) - u(\gamma(0)) = u(z) - u(0)$ is independent of the choice of γ, and $\int_\gamma \big(-u_y dx + u_x dy\big)$ is also independent of the choice of γ, because the one form $-u_y dx + u_x dy$ is closed due to the harmonicity of u: $d\big(-u_y dx + u_x dy\big) = (u_{xx} + u_{yy})dx \wedge dy = 0$, and the integration is done along paths confined in the disc D; or in more elementary language, the vector field $(-u_y, u_x)$ is curl free in D. Green's theorem then implies that the integral of $-u_y dx + u_x dy$ along closed simple piecewise curves in D is zero, which implies that $\int_{\gamma_1} \big(-u_y dx + u_x dy\big) = \int_{\gamma_2} \big(-u_y dx + u_x dy\big)$ if γ_1 and γ_2 are two simple curves in D from 0 to z that do not intersect each other. To allow more general curves, we define $v(z) = \int_{[0\ z]} \big(-u_y dx + u_x dy\big)$, where $[0\ z]$ is taken to be the radial path from 0 to z. Then Green's theorem implies that for h real, small, $v(z+h) - v(z) = \int_{[z\ z+h]} -u_y dx$, so $v_x = -u_y$, and similarly $v_y = u_x$. So for any curve γ from 0 to z, $\int_\gamma \big(-u_y dx + u_x dy\big) = \int_\gamma v_x dx + v_y dy = v(z) - v(0)$, which is independent of the choice of γ.)

To show that F is holomorphic we show that it has derivative f:

$$\lim_{\zeta\to 0}\frac{F(z+\zeta)-F(z)}{\zeta}-f(z)=\lim_{\zeta\to 0}\frac{1}{\zeta}\int_{\gamma'}(f(z')-f(z))\,dz'=0,$$

where γ' is the straight line from z to $z+\zeta$, and we have used

$$\left|\int_{\gamma'}(f(z')-f(z))\,dz'\right|\leq\max_{0\leq t\leq 1}|f(z+t\zeta)-f(z)||\zeta|,$$

and the continuity of f at z.

(c) $\mathrm{Re}(F(z))=Re\left(\int_\gamma(u_x-iu_y)(dx+idy)\right)=\int_\gamma u_x dx+u_y dy$
$=\int_\gamma\nabla u\cdot\vec{d\gamma}=u(\gamma(1))-u(\gamma(0))=u(z)-u(0).$

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $f(z)$ be a holomorphic function on the punctured disk

$$D_0 = \{z : 0 < |z| < 1\}.$$

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurent expansion of $f(z)$.

(a) Prove that for any $0 < r < 1$,

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n}.$$

[This is an instance of Parseval's theorem, which you may not quote.]

(b) Prove that if $\int_{D_0} |f(z)|^2 dA < \infty$, then $f(z)$ has a removable singular point at 0. Here dA is the Euclidean area element in $\mathbb{R}^2$.

SOLUTION: (a) For any $0 < r < 1$, $f(z) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta}$ is convergent uniformly and absolutely on the circle $S_r = \{|z| = r\}$. Thus by the Cauchy Multiplication Theorem we have

$$f(z) \cdot \overline{f(z)} = \sum_{n=-\infty}^{\infty} \sum_{k+l=n} a_k a_l r^{k+l} e^{i(k-l)\theta}$$

with the series on the right hand side converging uniformly and absolutely on S_r. Hence $\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 r^{2n}$.

(b). For any negative integer n, since

$$\int_D |f(z)|^2 dA = \int_0^1 r dr \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \geq 2\pi \int_0^1 \frac{|a_n|^2}{r} dr,$$

and $\int_0^1 \frac{1}{r} dr = \infty$, this forces $a_n = 0$ for all such n, therefore f has a removable singular point at 0.

5. Suppose f and g are holomorphic in a region containing the closed unit disc $\overline{D} = \{z : |z| \leq 1\}$. Suppose that f has a simple zero at $z = 0$ and vanishes nowhere else in $\overline{D}$. Let

$$f_t(z) = f(z) + tg(z).$$

Show that if $t > 0$ is sufficiently small, then

(a) $f_t(z)$ has a unique zero in $\overline{D}$, and

(b) if z_t is this zero, then the mapping $t \to z_t$ is continuous.

SOLUTION: For (a), since f vanishes only at $z = 0$ in $\overline{D}$, and $|f(z)|$ is continuous on $\partial\overline{D}$, we have $\min_{\partial\overline{D}}|f| > 0$. Thus, there exists $m > 0$ such that $|f(z)| \geq m > 0$ for all $z \in \partial\overline{D}$. Then for t small enough, we have $|f_t(z) - f(z)| = |t||g(z)| < m \leq |f(z)|$ for all $z \in \partial\overline{D}$, so Rouché's Theorem applies, giving f_t the same number of zeros in D as f. Since f has a single simple zero in D, this implies that f_t also has a single simple zero in D.

For (b), let $\epsilon > 0$ be given and t_0 sufficiently small that f_{t_0} has a unique zero in D at z_{t_0}. Claim: there is a $\delta > 0$ so that, for all t with $|t - t_0| < \delta$, we have $|z_{t_0} - z_t| < \epsilon$. To show this, consider

$$N_t^{(t_0)} = \frac{1}{2\pi i}\int_{C_{t_0}} \frac{f_t'(z)}{f_t(z)}dz,$$

where C_{t_0} is a circle of radius ϵ (small enough to remain in D) about z_{t_0}. By the argument principle, $N_t^{(t_0)}$ is the number of zeroes of f_t inside C_{t_0}, and clearly $N_{t_0}^{(t_0)} = 1$, since z_{t_0} is the unique zero of f_{t_0} in all of D. Since $|f_{t_0}|$ is continuous and non-vanishing on C_{t_0}, which is compact, it is bounded below by some $m > 0$. Likewise, $|g|$ is bounded above, on all of $\overline{D}$, by M, say. Choose $\delta < m/M$. Then for t sufficiently near t_0, $|t - t_0| < \delta$, we get that

$$|f_t| \; > \; |f_{t_0}| - |t - t_0||g| \; \geq \; m - \delta M \; > \; 0$$

is also bounded away from zero on C_{t_0}, so the integral makes sense. The integrand is jointly continuous in $(t, z) \in (t_0 - \delta, t_0 + \delta) \times C_{t_0}$, so $N_t^{(t_0)}$ is continuous in t, hence constant (since it is integer valued). So $N_t^{(t_0)} = N_{t_0}^{(t_0)} = 1$; that is, z_t is within ϵ of t_0, as claimed.

Another proof for (b) is via the Implicit Function Theorem. Set $F(z, t) = f_t(z) = f(z) + tg(z)$. Then $F(z, t)$ is continuously differentiable in z, with

$F(0,0) = 0$, and $F_z(0,0) = f'(0) \neq 0$–this is because $z = 0$ is a simple zero of f. Then the Implicit Function Theorem implies the existence of $\delta > 0$ and $r > 0$, as well as a continuous map $s : t \in (-\delta, \delta) \mapsto z = s(t) \in D_r(0)$ such that $F(s(t), t) = 0$ for all $t \in (-\delta, \delta)$. The Implicit Function Theorem also implies that for $t \in (-\delta, \delta)$, $z = s(t)$ is the only solution to $F(z,t) = 0$ in $D_r(0)$, agreeing with the solution in D identified in part (a)*. The same argument applies to any simple zero z_0 of $F(z_0, t_0)$.*

End of Session 2

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2016

Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

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Part I. Answer all questions.

1. Let X denote the set of all continuous real-valued functions $f : [0,1] \to \mathbb{R}$. For $f, g \in X$, define

$$d(f,g) = \max\Big\{\, |f(x) - g(x)| : 0 \le x \le 1 \,\Big\}\,.$$

a. ***Prove*** that d is a metric on X and that (X, d) is a complete metric space.

b. Let $\mathbf{0}$ denote the function in X which is identically equal to zero, and let $B = \{f \in X : d(f, \mathbf{0}) \le 1\}$. ***Prove*** that B is not compact. HINT: In a metric space, compactness is equivalent to sequential compactness.

SOLUTION: Let $f, g \in X$. Then the maximum

$$\max\Big\{\, |f(x) - g(x)| : 0 \le x \le 1 \,\Big\}$$

exists because $|f - g|$ is a continuous function on the compact interval $[0,1]$. *Furthermore, it is clear that*

$$d(f,g) \ge 0\,, \tag{1}$$

because all the numbers $|f(x) - g(x)|$ are ≥ 0.

It is clear from the definition of $d(f,g)$ that we have

$$d(g,f) = d(f,g)\,. \tag{2}$$

Let $f, g \in X$ be such that $d(f,g) = 0$. Then $|f(x) - g(x)| \le d(f,g) = 0$ for each x, so $f(x) = g(x)$ for each x. Hence

$$d(f,g) = 0 \Longrightarrow f = g\,. \tag{3}$$

Finally, let $f, g, h \in X$. Let $r = d(f,g)$, $s = d(g,h)$. Then, for each x, $|f(x) - h(x)| = |f(x) - g(x) + g(x) - h(x)| \le |f(x) - g(x)| + |g(x) - h(x)| \le r + s$. Therefore $\max\Big\{\, |f(x) - h(x)| : 0 \le x \le 1 \,\Big\} \le r + s$, so

$$d(f,h) \le d(f,g) + d(g,h)\,. \tag{4}$$

Formulas (1), (2), (3) and (4) prove that X is a metric space.

To prove that X is complete, we assume that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence with respect to the metric of X, and prove that $f_n \to f$ as $n \to \infty$ for some $f \in X$.

For each positive number ε, choose a positive integer $N(\varepsilon)$ such that $d(f_n, f_m) < \varepsilon$ whenever $n \geq N(\varepsilon)$ and $m \geq N(\varepsilon)$.

Then, if $x \in [0,1]$, the inequalities $|f_n(x) - f_m(x)| < \varepsilon$ hold for each n, m such that $n \geq N(\varepsilon)$ and $m \geq N(\varepsilon)$, because $d(f_n, f_m) < \varepsilon$ and $|f_n(x) - f_m(x)| \leq d(f_n, f_m)$. It follows that the numerical sequence $(f_n(x))_{n=1}^{\infty}$ is a Cauchy sequence. Hence the limit $f(x)$ of this sequence exists.

Given ε, since the inequality $|f_m(x) - f_n(x)| < \varepsilon$ holds for every m, n such that $m \geq N(\varepsilon)$ and $n \geq N(\varepsilon)$, we can let n go to infinity and conclude that $|f(x) - f_n(x)| \leq \varepsilon$ for all n such that $n \geq N(\varepsilon)$ and all x. This clearly implies that $f_n \to f$ uniformly, and from this it follows that f is continuous. (Proof: Fix $x \in [0,1]$. Given a positive ε, pick n such that $n \geq N\left(\frac{\varepsilon}{3}\right)$. Since f_n is continuous, we may pick a positive δ such that $|f_n(y) - f_n(x)| < \frac{\varepsilon}{3}$ whenever $|y - x| < \delta$. Then, if $|y - x| < \delta$, we have

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &\leq \varepsilon\,. \end{aligned}$$

This proves the continuity of f.)

Since f is continuous, f belongs to X. Since $|f(x) - f_n(x)| \leq \varepsilon$ for all n such that $n \geq N(\varepsilon)$, it follows that $d(f, f_n) \leq \varepsilon$ for all n such that $n \geq N(\varepsilon)$. Hence the sequence $(f_n)_{n=1}^{\infty}$ converges to f in X, and this proves that X is complete.

To prove that the closed unit ball $\mathcal{B}$ is not compact, it suffices by the hint to exhibit a sequence $(f_n)_{n=1}^{\infty}$ of members of $\mathcal{B}$ that does not have a convergent subsequence. Define $f_n(x) = x^n$, for $x \in [0,1]$ and n a positive integer. Then $f_n \in X$. If there was a subsequence $(f_{n_k})_{k=1}^{\infty}$ that converges in X to an $f \in X$, then this subsequence would converge to f uniformly, so it would also converge pointwise. But $(f_n)_{n=1}^{\infty}$ converges pointwise to the function g given by $g(x) = 0$ if $x < 1$, $g(1) = 1$. So the subsequence $(f_{n_k})_{k=1}^{\infty}$ would also converge pointwise to g. Hence $g = f$, contradicting the fact that g is discontinuous whereas f is continuous.

2. Let a, b be real numbers such that $a < b$, and let $f : [a,b] \to \mathbb{R}$.

a. ***Define*** what it means for f to be "absolutely continuous on $[a, b]$" (this is an ε-δ definition).

b. State a theorem relating the absolute continuity of such a function to its differentiability.

c. Assume that the restriction $f\Big|_{[\varepsilon,1]}$ is absolutely continuous for every ε such that $0 < \varepsilon < 1$, and that $\int_0^1 x^2 |f'(x)|^p dx < \infty$ for some real number p such that $p > 3$. Prove that $\lim_{x\to 0} f(x)$ exists and is finite. (HINT: Prove that $\int_0^1 |f'(x)|dx < \infty$.)

SOLUTION: (a). f is absolutely continuous on $[a, b]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that if the finite sequence $\big((x_k, y_k)\big)_{k=1}^N$ of pairwise disjoint sub-intervals of $[a, b]$ satisfies $\sum_k (y_k - x_k) < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \epsilon$.

(b). Theorem: A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ if and only if it is differentiable almost everywhere, the derivative f' is Lebesgue integrable, and $f(y) - f(x) = \int_x^y f'(u)du$ for all x, y such that $a \le x \le y \le b$.

(c). Assume that the conditions of Part c hold. Then $f'(x)$ exists for almost every $x \in [0, 1]$. Furthermore, for each ε such that $0 < \varepsilon < 1$, we have

$$f(1) - f(\varepsilon) = \int_\varepsilon^1 f'(x)dx\,,$$

so

$$f(\varepsilon) = f(1) - \int_\varepsilon^1 f'(x)dx\,,$$

If we can prove that f' is integrable on $[0, 1]$, then it will follow that the limit $\lim_{x\to 0} f(x)$ exists and is finite, because, if $f' \in L^1([0, 1])$, then

$$\begin{aligned} \lim_{x\to 0} f(x) &= \lim_{x\to 0} \Big(f(1) - \int_x^1 f'(u)du \Big) \\ &= f(1) - \int_0^1 f'(u)du\,. \end{aligned}$$

So it suffices to prove that $\int_0^1 |f'(x)|dx < \infty$.

Now, if we let $g(x) = x^{\frac{2}{p}} |f'(x)|$, $h(x) = x^{-\frac{2}{p}}$, we have

$$|f'(x)| = g(x)h(x)\,,$$

so by Hölder's inequality

$$\begin{aligned}\int_0^1 |f'(x)|dx &= \int_0^1 g(x)h(x)dx \\ &\leq \Big(\int_0^1 g(x)^p dx\Big)^{\frac{1}{p}}\Big(\int_0^1 h(x)^q dx\Big)^{\frac{1}{q}} \\ &= \Big(\int_0^1 x^2|f'(x)|^p dx\Big)^{\frac{1}{p}}\Big(\int_0^1 x^{-\frac{2q}{p}}|dx\Big)^{\frac{1}{q}},\end{aligned}$$

where q is the conjugate exponent of p, characterized by $\frac{1}{p}+\frac{1}{q}=1$. Since we are assuming that $\int_0^1 x^2|f'(x)|^p dx < \infty$, it suffices to show that $\int_0^1 x^{-\frac{2q}{p}}dx < \infty$.

Since $\frac{1}{p}+\frac{1}{q}=1$, we have $\frac{p}{p}+\frac{p}{q}=p$, i.e. $\frac{p}{q}=p-1$. Since $p>3$, it follows that $\frac{p}{q}>2$, so $\frac{p}{2q}>1$, and then $\frac{2q}{p}<1$. Hence the function $[0,1]\ni x\mapsto x^{-\frac{2q}{p}}dx<\infty$ is integrable, so the integral $\int_0^1 x^{-\frac{2q}{p}}dx$ is finite. As explained in the previous paragraph, this proves our result.

3. Let n be a positive integer.

a. ***Define*** what it means for a subset S of $\mathbb{R}^n$ to be "connected".

b. Let Ω be an open connected subset of $\mathbb{R}^n$. Let $f:\Omega\to\mathbb{R}$ be a function such that

$$\lim_{\varepsilon\to 0}\frac{f(p+\varepsilon v)-f(p)}{\varepsilon}=0$$

for every $p\in\Omega$ and every $v\in\mathbb{R}^n$. ***Prove*** that f is a constant. Make sure that you use in this proof the definition of "connected" that you gave in Part a.

SOLUTION: (a). A subset S of $\mathbb{R}^n$ is connected if it is not possible to express S as the union of two nonempty subsets S_1, S_2 of S such that (a) $S_1\cap S_2=\emptyset$, and (b) S_1 and S_2 are relatively open subsets of S. (A subset T of S is relatively open in S if $T=U\cap S$ for some open subset U of $\mathbb{R}^n$.)

(b). Suppose that Ω is an open connected subset of $\mathbb{R}^n$. Let f be a function that satisfies the condition of Part b.

If p,q are two points of $\mathbb{R}^n$, the segment from p to q is the set

$$\sigma_{p,q}=\{(1-t)p+tq : 0\leq t\leq 1\}.$$

We prove that

(*) f is constant on every segment $\sigma_{p,q}$ such that $\sigma_{p,q} \subseteq \Omega$.

To prove (*), fix p and q, and define $g(t) = f((1-t)p + tq)$, for $0 \le t \le 1$. Then g is a function of one variable, defined on the interval $[0,1]$. For each $t \in [0,1]$, the limit

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$

exists and is equal to zero. (Reason: if we let $A = (1-t)p+tq$ and $B = q-p$, then

$$\begin{aligned} g(t+h) - g(t) &= f((1-t-h)p + (t+h)q) - f((1-t)p + tq) \\ &= f(A + hB) - f(A)\,, \end{aligned}$$

so $\lim_{h\to 0} \frac{g(t+h)-g(t)}{h} = \lim_{h \to 0} \frac{f(A+hB)-f(A)}{h} = 0$.) So g is a function of $[0,1]$ that has a derivative equal to zero everywhere. Hence, by the Mean Value Theorem for functions of one variable, g is a constant function. Clearly, this implies that f is constant on the segment $\sigma_{p,q}$, and (*) is proved.

Now, if D is any disc which is contained in Ω, the function f must be constant on D, because, if p is the center of D, and q is any point of D, the segment $\sigma_{p,q}$ is contained in D, so f is constant on $\sigma_{p,q}$, so $f(q) = f(p)$.

We now prove, finally, that f is constant on Ω.

For each real number c, let $S_c = \{p \in \Omega : f(p) = c\}$. Then we have proved that every set S_c is open. Pick a point p_0 of Ω, and let $c_0 = f(p_0)$. Let $A = S_{c_0}$, $B = \bigcup_{c \ne c_0} S_c$. Then both A and B are open, and obviously $A \cap B = \emptyset$ and $A \cup B = \Omega$. Furthermore, A is nonempty, because $p_0 \in A$. Since Ω is connected, B must be empty, so $\Omega = A$, which means that f is constant on Ω.

A different definition for connectedness may be used, but the argument for part (b) must use the same definition for connectedness.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that

$$\int_{-\infty}^{\infty} (1 + |x|)|f(x)|dx < \infty\,.$$

Define

$$g(y) = \int_{-\infty}^{\infty} f(x)\cos(xy)dx\,.$$

1. ***Prove*** that g is continuously differentiable (that is, prove that the derivative $g'(y) = \lim_{h\to 0} \frac{g(y+h)-g(y)}{h}$ exists for every $y \in \mathbb{R}$, and is a continuous function of y).

2. ***Write a formula*** for g', as an integral.

SOLUTION: Fix a real number y. If $h \in \mathbb{R}$ and $h \neq 0$, we have

$$g(y+h) - g(y) = \int_{-\infty}^{\infty} f(x)\Big(\cos(x(y+h)) - \cos(xy)\Big)dx$$

so

$$\frac{g(y+h)-g(y)}{h} = \int_{-\infty}^{\infty} k_h(x)dx\,,$$

where

$$k_h(x) = f(x)\frac{\cos(x(y+h)) - \cos(xy)}{h}\,.$$

For each fixed x the limit

$$\lim_{h\to 0}\left(f(x)\frac{\cos(x(y+h)) - \cos(xy)}{h}\right)$$

is the derivative of the function $\mathbb{R} \ni u \mapsto f(x)\cos(xu)$ at $u = y$, which is equal to $-xf(x)\sin xy$. So, if we let $\hat{k}(x) = -xf(x)\sin xy$, the functions k_h converge pointwise to $\hat{k}$ as $h \to 0$. The assumption that $\int_{-\infty}^{\infty}(1+|x|)|f(x)|dx < \infty$ implies that $\hat{k}$ is integrable. We can apply Lebesgue dominated convergence theorem to conclude

$$g'(y) = \lim_{h\to 0}\frac{g(y+h)-g(y)}{h} = -\int_{-\infty}^{\infty} xf(x)\sin(xy)dx\,. \tag{5}$$

if we find an integrable function $\kappa : \mathbb{R} \to \mathbb{R}$ such that $|k_h(x)| \leq \kappa(x)$ for all $x \in \mathbb{R}$ and all small h. (Precisely: Suppose such a function κ exists. To prove that $\lim_{h\to 0}\int_{-\infty}^{\infty} k_h(x)dx = \int_{-\infty}^{\infty}\hat{k}(x)dx$, it suffices to show that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} k_{h_n}(x)dx = \int_{-\infty}^{\infty}\hat{k}(x)dx \tag{6}$$

for every sequence $(h_n)_{n=1}^{\infty}$ of nonzero real numbers that converges to 0 as $n \to \infty$. If $(h_n)_{n=1}^{\infty}$ is such a sequence, then $\lim_{n\to\infty} k_{h_n}(x) = \hat{k}(x)$ for every x, and in addition $|k_{h_n}(x)| \le \kappa(x)$ for every x and every n, so the dominated convergence theorem applies and we can conclude that (6) holds.)

For each fixed x, we have

$$\frac{d}{du}\cos(xu) = -x\sin(xu)\,.$$

It then follows from the Mean Value Theorem that

$$\cos(x(y+h)) - \cos(xy) = -x\sin(xc)h \qquad \text{for some } c \in \mathbb{R}\,.$$

Therefore

$$\left|\frac{\cos(x(y+h)) - \cos(xy)}{h}\right| \le |x|\,,$$

so $|k_h(x)| \le |xf(x)|$ for all x and all h, Hence, if we define κ by letting $\kappa(x) = |xf(x)|$, we have shown that $|k_h(x)| \le \kappa(x)$ for all x and all h, Since we are assuming that κ is integrable, Formula (5) is now rigorously justified.

Finally, it follows clearly from (5) that the derivative g' is continuous. Indeed: if $(y_n)_{n=1}^{\infty}$ is a sequence of real numbers that converges to a limit $y \in \mathbb{R}$, then the functions $\mathbb{R} \ni x \mapsto xf(x)\sin(xy_n)$ converge pointwise to the function $\mathbb{R} \ni x \mapsto xf(x)\sin(xy)$, and are uniformly dominated by the integrable function κ. So, by the dominated convergence theorem, $\lim_{n\to\infty} g'(y_n) = g'(y)$.

5. Let T be a real number such that $T > 0$. Let $f : (0,T) \to \mathbb{R}$ be a Lebesgue integrable function. (Here $(0,T)$ is the open interval $\{x \in \mathbb{R} : 0 < x < T\}$.) Define a function $g : (0,T) \to \mathbb{R}$ by letting

$$g(x) = \int_x^T \frac{f(t)}{t}\,dt\,.$$

Prove that g is integrable on $(0,T)$ and $\int_0^T g(x)dx = \int_0^T f(x)dx$.
HINTS: (a) You may want to consider first the case in which f is nonnegative. (b) Use the Fubini-Tonelli theorem.)

SOLUTION: It suffices to assume that f is nonnegative. (Reason: Suppose the conclusion is true for nonnegative f. Let $f : (0,T) \to \mathbb{R}$ be an arbitrary integrable function. Then we can write $f = f_+ - f_-$, with f_+, f_- nonnegative *and integrable. If we let* $g_+(x) = \int_x^T \frac{f_+(t)}{t}dt$ *and* $g_-(x) = \int_x^T \frac{f_-(t)}{t}dt$*, then* $g = g_+ - g_-$*, and* g_+*,* g_- *are both integrable by our assumption, so* g *is integrable. Furthermore, our assumption, also implies that* $\int_0^T g_+(x)dx = \int_0^T f_+(x)dx$ *and* $\int_0^T g_-(x)dx = \int_0^T f_-(x)dx$*. So* $\int_0^T g(x)dx = \int_0^T f(x)dx$*.)*

Assume now that f *is nonnegative, so* g *is nonnegative as well. Furthermore,* g *is clearly measurable. (Actually,* g *is continuous on the interval* $(0, T]$*.) So, if we prove that*

$$\int_0^T g(x)dx = \int_0^T f(x)dx\,, \tag{7}$$

then it will follow that $\int_0^T g(x)dx < \infty$ *(because we are assuming that* f *is integrable, so* $\int_0^T f(x)dx < \infty$*), and this implies that* g *is integrable as well.*

For $0 \le x \le T$*,* $0 \le t \le T$*, define*

$$h(x,t) = \begin{cases} \frac{f(t)}{t} & \text{if} \quad x < t \\ 0 & \text{if} \quad x \ge t\,. \end{cases}$$

Then h *is a nonnegative real-valued measurable function on the product* $P = [0, T] \times [0, T]$*. So, by the Fubini-Tonelli theorem, the double integral*

$$I = \iint_P h\,dm_2$$

(where m_2 *is 2-dimensional Lebesgue measure) and the two iterated integrals*

$$I_1 = \int_0^T \left(\int_0^T h(x,t)dt\right)dx,$$

$$I_2 = \int_0^T \left(\int_0^T h(x,t)dx\right)dt$$

are equal. So in particular $I_1 = I_2$*.*

Now, we have, for each x*,*

$$\begin{aligned} \int_0^T h(x,t)dt &= \int_x^T \frac{f(t)}{t}dt \\ &= g(x)\,. \end{aligned}$$

So $I_1 = \int_0^T g(x)dx$.

On the other hand, we have, for each t *such that* $t > 0$,

$$\begin{aligned} \int_0^T h(x,t)dx &= \int_0^t \frac{f(t)}{t} dx \\ &= t \cdot \frac{f(t)}{t} \\ &= f(t)\,, \end{aligned}$$

so $I_2 = \int_0^T f(t)dt$.

Since $I_1 = I_2$, *(7) follows, and our proof is complete.*

End of Session 3