RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS
Written Qualifying Examination

Fall 2002, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. A metric space X is called *separable* if X has a countable dense subset. Prove that every non-empty subset of a separable metric space is separable.

2. Evaluate the integral

$$\int_0^\infty \frac{\cos x}{x^2+1}\, dx\ .$$

3. Suppose that A is an $n \times n$ real orthogonal matrix and that $\det(A) = -1$.

a) Prove that -1 is an eigenvalue of A.

b) If $n = 3$ and 1 is also an eigenvalue of A, prove that A is symmetric.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Prove that an entire function whose range is contained in the upper half plane must be constant.

5. Let X be a compact metric space with the property that for every $\epsilon > 0$ and for any two points $x, y \in X$, there is an integer N and a finite sequence of points $\{x_i\}_{i=0}^{N+1}$ such that $x_0 = x$, $x_{N+1} = y$, and $d(x_i, x_{i+1}) < \epsilon$ for all i with $0 \leq i \leq N$. Prove that X is connected.

6. Suppose n is a positive integer and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a map of class C^1. That is, the partial derivatives $\frac{\partial f_j}{\partial x_i}(x)$ of the components $(f_1, \ldots, f_n)$ of f exist at every point $x \in \mathbb{R}^n$ and are continuous functions of x. Prove that if E is a subset of $\mathbb{R}^n$ such that $\text{meas}(E) = 0$ then $\text{meas}(f(E)) = 0$. Here "meas" denotes Lebesgue measure.

7. Let $f(x) = x^4 + 1$.

a) If $\mathbb{Q}$ is the field of rational numbers, prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

b) Suppose that p is any prime number and $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ is the finite field with p elements. Prove that $f(x)$ is always reducible in $\mathbb{F}[x]$ for all primes p.

8. a) Prove that if f is entire and that $f(z) \neq 0$ for all $z \in \mathbb{C}$ then $f = e^g$ where g is also entire.

b) Assume f is entire, nonconstant, and that $|f(z)| \leq e^{\sqrt{|z|}}$ for all $z \in \mathbb{C}$. Prove that *either f is a polynomial or* it has infinitely many zeros in $\mathbb{C}$.

The italicized words weren't on the actual exam: only a tiny discrepancy: apologies from the Examining Committee.

9. Let p be an odd prime, and let R be the ring of "p-quaternions"; that is

$$R = \{\alpha_0 + \alpha_1 i + \alpha_2 j + \alpha_3 k : \alpha_n \in \mathbb{Z}/p\mathbb{Z} \text{ for } n = 0, 1, 2, 3, \text{ and } \\ i^2 = j^2 = k^2 = -1, ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik\}.$$

Prove that R is a simple ring, that is, the only two sided ideals of R are the zero ideal and R itself.

RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Fall 2002, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let X be a compact metric space and let A and B be disjoint closed subsets of X. Prove that there exist disjoint open subsets U and V of X with $A \subseteq U$ and $B \subseteq V$.

2. Prove that

$$\lim_{\alpha \to +\infty} \int_0^{+\infty} e^{-x^4} \sin(\alpha x^2)\, dx = 0\,.$$

3. Suppose that A and B are $n \times n$ complex nilpotent matrices with the same rank and the same minimal polynomial.

a) If $n = 6$, prove that A and B are similar.

b) If $n = 7$, is this still true? Either prove or give a counterexample.

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let X be a separable metric space, and let $f\colon X \times \mathbb{R} \to \mathbb{R}$ be a function such that

   i) The function $X \ni x \mapsto f(x,r) \in \mathbb{R}$ is continuous for every $r \in \mathbb{R}$, and

   ii) The function $\mathbb{R} \ni r \mapsto f(x,r)$ is measurable for every $x \in X$.

Let $\varphi(x) = \inf\{f(x,r) : r \in \mathbb{R}\}$, so φ is a function from X to $\mathbb{R} \cup \{-\infty\}$. Prove that φ is measurable.

5. Let $V = \left\{ \begin{bmatrix} \lambda \\ \mu \\ \nu \end{bmatrix} : \lambda, \mu, \nu \text{ integers} \right\}$, so V is a free abelian group of rank 3.

Suppose that $a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$ and $b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$ are in V. Let U be the subgroup of V generated by a and b. Assume that U has rank 2.

State and prove a necessary and sufficient condition involving the three determinants $\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}$, $\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{vmatrix}$, and $\begin{vmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{vmatrix}$, insuring that V/U is free abelian of rank 1.

6. Prove that if f is an entire function and $|f(z)| \le e^{|z|}$ for all $z \in \mathbb{C}$ then $|f'(z)| \le e^{|z|+1}$ for all $z \in \mathbb{C}$.

7. Prove that the integral

$$\int_{-\infty}^{\infty} e^{-(t-i\gamma)^2}\, dt$$

is independent of the real parameter γ.

8. Construct an example of an open map between metric spaces which is not continuous.

9. If G is a group with order 2002, prove that G has a cyclic subgroup of index 2.