RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Fall 2003, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Let G be a nontrivial group. Show that G is simple if and only if the diagonal $D = \{(g, g) \mid g \in G\}$ is a maximal subgroup of $G \times G$.

2. Let C and R be positive numbers and n be a positive integer. Suppose that f is an entire function which satisfies $|f(z)| \geq C|z|^n$ for all $z \in \mathbb{C}$ with $|z| \geq R$. Show that f is a polynomial of degree at least n.

3. Suppose that E and F are Lebesgue measurable subsets of $\mathbb{R}$. Prove that $E \times F$ is a Lebesgue measurable subset of $\mathbb{R}^2$ and that $|E \times F| = |E||F|$, where $| \cdot |$ denotes two-dimensional Lebesgue measure on the left hand side and one dimensional Lebesgue measure on the right. (Interpret $0 \cdot \infty$ as 0 in this equation.)

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First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let r and n be positive integers, let G be a group generated by r elements, and let S be the set of subgroups of G with index at most n.

(a) Show that S is finite.

(b) Suppose that $r = 2$ and $n = 10$. Give an upper bound for the cardinality of S.

5. Find the fractional linear transformation T which maps the circle $|z| = 2$ onto the circle $|z + 1| = 1$ in the complex plane and that maps the points -2 and 0 onto 0 and 1 respectively.

6. If $a > 0$ and $\xi \in (-\infty, +\infty)$, show that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2 + x^2}\, e^{-2\pi i x \xi}\, dx = e^{-2\pi a|\xi|}.$$

7. Let X be a metric space. Assume that $x \in X$ is a point such that $X \backslash \{x\}$ is compact. Prove:

(a) The one-element set $\{x\}$ is open in X.

(b) The subset $\{x\}$ is a connected component of X.

Exam continues on next page

For **Questions 8** and **9** below, we let $L^p(\mathbb{R}, dx)$, $p \geq 1$, denote the Banach space of (equivalence classes of) functions $f : \mathbb{R} \to \mathbb{C}$ which are power-p-integrable with respect to Lebesgue measure dx on $\mathbb{R}$, and let

$$\|f\|_{L^p(dx)} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p} < \infty,$$

denote the Banach space norm of f. In particular, $L^1(\mathbb{R}, dx)$ is the Banach space of Lebesgue integrable functions, and $L^2(\mathbb{R}, dx)$ the Hilbert space of square-integrable functions. If $k \in \mathbb{R}$ is used instead of $x \in \mathbb{R}$, we indicate this by $L^2(\mathbb{R}, dk)$ and denote the norm of f by $\|f\|_{L^2(dk)}$.
We also let $\tau_y : L^2(dx) \to L^2(dx)$ denote the translation operator acting on f, thus $\tau_y f(x) = f(x - y)$. We say that a function $f \in L^2(\mathbb{R}, dx)$ has an L^2 *derivative* if there exists a $g \in L^2(\mathbb{R}, dx)$ such that

$$\lim_{y \to 0} \left\| \frac{\tau_{-y} f - f}{y} - g \right\|_{L^2(dx)} = 0$$

and in this case we write f' for g.
By $\mathcal{F} f \equiv \hat{f} \in L^2(\mathbb{R}, dk)$ we denote the *Fourier transform* of $f \in L^2(\mathbb{R}, dx)$. Recall that for $f \in (L^1 \cap L^2)(\mathbb{R}, dx)$ the Fourier transform is defined by

$$(\mathcal{F} f)(k) = \hat{f}(k) = \int_{\mathbb{R}} f(x) e^{-i2\pi kx} \, dx$$

and that by Plancherel's theorem $\mathcal{F}$ extends uniquely to a unitary isomorphism between $L^2(\mathbb{R}, dx)$ and $L^2(\mathbb{R}, dk)$.

8. *Please see the preceding paragraph for an explanation of notation and terminology.*
Suppose that $f \in L^2(\mathbb{R}, dx)$ has the properties
(i) $f \in L^2(dx)$, (ii) $xf \in L^2(\mathbb{R}, dx)$, and (iii) $k\hat{f} \in L^2(\mathbb{R}, dk)$.

Prove that f', the L^2 derivative of f, exists, and that the Fourier transforms of f and f' are related by

$$\widehat{f'}(k) = 2\pi i k \hat{f}(k).$$

(**Hints:** Work out the Fourier transform of a translate of f. Make use of Plancherel's theorem. Use a Taylor's remainder estimate for $e^{i2\pi ky} - 1$.)

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For **Question 9** below, recall that a functional $F : L^2(\mathbb{R}, dx) \to \mathbb{R}$ is *norm continuous* if $|F(g) - F(f)| \to 0$ whenever $\|g - f\|_{L^2(dx)} \to 0$, and F is *weakly lower semi-continuous* if

$$\liminf_{j\to\infty} F(g_j) \geq F(g)$$

for all $g \in L^2(\mathbb{R}, dx)$ and sequences $\{g_j\}_{j\in\mathbb{N}} \in L^2(\mathbb{R}, dx)$ *converging weakly* to g, that is,

$$\lim_{j\to\infty} \int_{\mathbb{R}} hg_j\, dx = \int_{\mathbb{R}} hg\, dx, \quad \text{for all } h \in L^2(\mathbb{R}, dx).$$

9. *Please see the paragraphs preceding this question and Question 8 for an explanation of notation and terminology.*
Let B and S denote the norm-closed ball and sphere in $L^2(\mathbb{R}, dx)$, respectively, both with radius one and center at the origin:

$$B = \{f \in L^2(\mathbb{R}, dx) : \|f\|_{L^2} \leq 1\},$$
$$S = \{f \in L^2(\mathbb{R}, dx) : \|f\|_{L^2} = 1\}.$$

Choose $f_* \in S$ and define a functional $F : L^2(\mathbb{R}, dx) \to \mathbb{R}$ by setting

$$F(f) = \frac{1}{1 + \|f - f_*\|_{L^2}^2}.$$

Prove:

(a) F is norm continuous but not weakly lower semi-continuous. (**Hint:** Recall Fatou's lemma.)

(b) $F(f)$ achieves its minimum value on B at $f = -f_*$.

Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Fall 2003, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Let A be a 3×3 complex matrix. Let $C(A)$ be the vector space of complex matrices that commute with A. Show that the complex dimension of $C(A)$ is at least 3.

In **Question 2** below, we let $L^2(\mathbb{R}, dx)$ denote the Hilbert space of functions $f : \mathbb{R} \to \mathbb{C}$ which are square-integrable with respect to Lebesgue measure dx on $\mathbb{R}$, so

$$\|f\|^2_{L^2(dx)} = \int_{\mathbb{R}} |f(x)|^2\, dx < \infty,$$

where $\|f\|_{L^2(dx)}$ denotes the Hilbert space norm of f. Recall that $f \in L^2(\mathbb{R}, dx)$ is the *strong limit* of a sequence $\{f_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}, dx)$ if

$$\lim_{j\to\infty} \|f - f_j\|_{L^2(dx)} = 0,$$

while $f \in L^2(\mathbb{R}, dx)$ is the *weak limit* of a sequence $\{f_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{R}, dx)$ if

$$\lim_{j\to\infty} \int_{\mathbb{R}} g f_j\, dx = \int_{\mathbb{R}} g f\, dx.$$

for every $g \in L^2(\mathbb{R}, dx)$.

2. Let $\{e_j \in L^2(\mathbb{R}, dx)\}_{j\in\mathbb{N}}$ be an orthonormal basis for $L^2(\mathbb{R}, dx)$. Prove:

(a) The strong limit of the sequence $\{e_j\}_{j\in\mathbb{N}}$ does not exist, as $j \to \infty$.

(b) The sequence $\{e_j\}_{j\in\mathbb{N}}$ has weak limit $0 \in L^2(\mathbb{R}, dx)$, as $j \to \infty$.

3. Let X, Y be metric spaces and let $f : X \to Y$ be a continuous map such that (i) f maps closed sets to closed sets and (ii) the inverse image of any point in Y is compact. Show that $f^{-1}(K)$ is compact whenever K is compact.

Exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $\mathrm{GL}(n, \mathbb{C})$ denote the multiplicative group of n-by-n nonsingular complex matrices.

(a) Show that an element of finite order in $\mathrm{GL}(n, \mathbb{C})$ is diagonalizable.

(b) Determine the number of conjugacy classes of elements of order 4 in $\mathrm{GL}(3, \mathbb{C})$.

5. Give an explicit description of one Sylow 3-subgroup of the symmetric group S_9.

6. Let $u(x, y)$ be harmonic and real-valued in the plane. Show that for all $r > 0$ and all (x, y),

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\, d\theta.$$

7. Let $P(z)$ be a polynomial in z of degree two or higher. Let $C_R = \{z \in \mathbb{C} : z = Re^{i\theta}, \theta \in (0, \pi)\}$ be the positively oriented semi-circle of radius R in the upper half plane, centered at the origin. Show that for all $k \geq 0$,

$$\lim_{R\to\infty} \int_{C_R} \frac{e^{ikz}}{P(z)} dz = 0.$$

8. For each integer $k > 0$ suppose that $f_k : (0, 1) \to \mathbb{R}$ is a nonnegative Lebesgue measurable function. Suppose that for some $p > 1$ there is a constant M such that

$$\int_0^1 (f_k(t))^p\, dt < M, \quad \text{for all } k > 0.$$

If b is any positive number, prove that there is a positive number a such that for any Lebesgue measurable subset $E \subset (0, 1)$ with measure $|E| < a$ we have

$$\int_E f_k(t)\, dt < b, \quad \text{for all } k > 0.$$

Exam continues on next page

9. Let $\mathbb{R}^2$ denote Euclidean space with its standard norm $\|\cdot\|$. Define a metric space (X, d) by setting $X = \mathbb{R}^2$ and $d(x_1, x_2) = \|x_1\| + \|x_2\|$ if x_1, x_2 are points in X which do not lie on the same ray from 0, and $d(x_1, x_2) = \|x_1 - x_2\|$ otherwise. (Think of this as the train metric: to get anywhere, you have to go into New York first, and then back out, unless your stop is on the same line.)

(a) Show that d is a metric.

(b) Let $B = \{x \in X : d(x, 0) \leq 1\}$ denote the unit ball centered at 0. Is B connected? Is B compact?

(c) Let $d'(x_1, x_2) = \|x_1\| + \|x_2\| + 1$ if x_1, x_2 are points in X which do not lie on the same ray from 0, and $d'(x_1, x_2) = \|x_1 - x_2\|$ otherwise. (Think of this as follows: passengers have to wait an hour at Penn Station). Is d' a metric?

Exam End