RUTGERS UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Fall 2004, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9).

If you work on four or more questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then only the first three questions attempted will be graded, as determined by the order in which they appear in the examination book(s).

First Day—Part I: Answer each of the following three questions

1. Let R be a finite associative ring with unit. Prove that any left zero divisor is a right zero divisor. (Recall that $r \in R$ is called a *left zero divisor* if $rs = 0$ for some nonzero $s \in R$, and a *right zero divisor* if $qr = 0$ for some nonzero $q \in R$.)

2. Let X be an open subset of the unit disk $\{\mathbf{x} : ||\mathbf{x}|| < 1\}$ in $\mathbb{R}^n$. Let Y denote the union of X and a point "∞" ($\infty \notin X$), endowed with the following topology:

> An open subset of Y is either an open subset of X, or else $U \cup \{\infty\}$ for some U open in X such that $X - U$ is closed in $\mathbb{R}^n$. (You may assume it is known that Y is a topological space.)

a) Show that Y is a Hausdorff topological space.

b) Show that Y is a compact topological space.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative and measurable function, and assume that both $\int_{\mathbb{R}} f(t)\, dt < \infty$ and $\int_{\mathbb{R}} e^t f(t)\, dt < \infty$.

Show that the integral $G(x) = \int_{\mathbb{R}} e^{tx} f(t)\, dt$ is finite when $0 \leq x \leq 1$. Then prove that the function $G(x)$ is continuous on $0 \leq x \leq 1$, and differentiable on $0 < x < 1$. Be sure to present your arguments in detail.

First Day—Part II: Answer three of the following six questions. If you work on four or more questions, indicate clearly which three should be graded.

4. Let μ and ν be finite measures on the Borel sets of $\mathbb{R}$ and let $\mu \otimes \nu$ denote the product measure. Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ denote $\Phi(x, y) = x + y$. Define the measure

$$\tau(A) = \mu \otimes \nu(\Phi^{-1}(A))$$

on the Borel sets of $\mathbb{R}$. Show that for every f such that $\int_{\mathbb{R}} |f| d\tau < \infty$:

$$\int_{\mathbb{R}} f\, d\tau = \int\int_{\mathbb{R}^2} f(x + y) d\mu(x) d\nu(x).$$

5. Prove that the commutative ring $\mathbb{R}[x]/(x^2 + x + 1)$ is isomorphic to $\mathbb{C}$.

6. Is the following statement true or false? Explain carefully.

> "If f is meromorphic, then so is the composition $f \circ f$."

7. Let G be the set of all complex 3×3 matrices which have exactly one nonzero element in every row and in every column. You may assume that G is a group under matrix multiplication. Show that G has two normal subgroups G_1 and G_2, with $G_1 \subset G_2 \subset G$, such that G_1, G_2/G_1, G/G_2 are abelian groups.

8. Find the number of roots of the polynomial $P(z) = z^5 + 2z^3 + 3$ in the *closed* unit disk $\{z : |z| \leq 1\}$.

9. Let E be a subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ be an absolutely continuous function. Prove that there exists a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that the restriction g on E equals f.

Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

Fall 2004, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9).

If you work on four or more questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then only the first three questions attempted will be graded, as determined by the order in which they appear in the examination book(s).

Second Day—Part I: Answer each of the following three questions

1. Let m be a Lebesgue measure on $\mathbb{R}$. Let A, B, E, F be subsets of $[0,1]$ such that $A \cup B = [0,1]$, $A \subset E$, $B \subset F$, E and F are Lebesgue measurable and $m(E) + m(F) = 1$. Prove that A and B are Lebesgue measurable.

2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of an $n \times n$-matrix A over $\mathbb{C}$. (Eigenvalues are listed by their multiplicity and may appear more than once — do not assume that the eigenvalues are distinct.) Prove that for any natural number k the trace of A^k is equal to

$$\lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k.$$

3. Assume that $f : \mathbb{C} \to \mathbb{C}$ is an entire function whose real part $\text{Re}f$ satisfies $\text{Re}f(z) > 0$ for all $z \in \mathbb{C}$. Can $f(z)$ be nonconstant? Explain carefully.

Second Day—Part II: Answer three of the following six questions. If you work on four or more questions, indicate clearly which three should be graded.

4. Let m be a Lebesgue measure on $\mathbb{R}$. Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function. Let $A \subset [0,1]$ satisfy $m(A) = 0$. Prove that $m(f(A)) = 0$.

5. Let G be a group of order p^n, where p is a prime and n is a positive integer. Prove that the center of G contains more than one element.

6. Let f be an analytic function on the domain $\{z : 0 < |z| < 1\}$, and that its real part satisfies $\liminf_{z \to 0} \text{Re}f(z) > -\infty$. Show that f has a removable singularity at zero.

7. Consider the vector space of polynomials $\mathbb{R}[x]$, made into a normed vector space with the norm $||P|| = \sum |c_i|$, $P(x) = \sum c_i x^i$. Determine whether or not the "unit cube" $X = \{P(x) : ||P|| \leq 1\}$ is compact. Prove or disprove this carefully.

8. Let $A = (a_{ij})$ be an $n \times n$ matrix of real numbers. Prove that

$$\det(A)^2 \leq \prod_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2.$$

9. Let $f : S^1 \to S^2$ be a continuous function which is not onto. Show that f extends to a continuous function F from the closed unit disk D in the plane to S^2 in the sense that the restriction of F to S^1 is f.

Exam End