RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2005, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

First Day—Part I: Answer each of the following three questions

1. Let $f : M_n(\mathbb{C}) \to \mathbb{C}$ be a linear functional on the vector space $M_n(\mathbb{C})$ of $n \times n$ matrices over the complex numbers $\mathbb{C}$. Assume that f satisfies the conditions $f(AB) = f(BA)$. Show that there exists a complex number λ such that $f(A) = \lambda(\text{Trace}A)$ for all $A \in M_n(\mathbb{C})$.

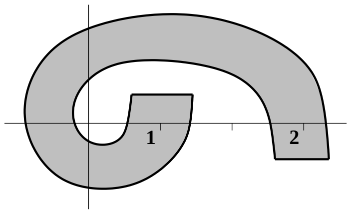
2. Suppose that f is a Lebesgue measurable function f on the square $[0,1] \times [0,1]$ such that both of the iterated integrals

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy, \qquad \int_0^1 \int_0^1 f(x,y)\,dy\,dx$$

make sense. Must these integrals be equal? Prove or give a counterexample.

3. a) Suppose $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, the punctured unit disc. Prove that there is no function $f(z)$ holomorphic in U^* so that $(f(z))^2 = z$.

b) Suppose that R is the domain drawn to the right. Prove that there is a function $g(z)$ holomorphic in R with $(g(z))^2 = z$ for all $z \in R$ and so that $g(1) = -1$. What is $g(2)$?



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First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let E be a measurable subset of $\mathbb{R}$. Suppose $f \in L^1(E)$ and $f(x) > 0$ for almost every $x \in E$. Prove that

$$\lim_{k\to\infty} \int_E [f(x)]^{1/k}\, dx = m(E).$$

5. Suppose that $A = \{a_n\}$ is a strictly increasing sequence of positive integers. Construct a function $f(z)$ which is holomorphic in $\mathbb{C} \setminus A$ and which has a simple pole at every $a_n \in A$.

6. Let $R = \mathbb{C}[x, x^{-1}, (x-1)^{-1}]$ be the ring consisting of rational functions of the form

$$\frac{g(x)}{x^i(x-1)^j}, \qquad i, j \geq 0 \text{ and } g(x) \in \mathbb{C}[x].$$

Show that R is a principal ideal domain.

7. Classify all groups of order 12. Justify your answer.

9. A continuous map is *proper* if the inverse of any compact set is compact.
a) If $f\colon \mathbb{C} \to \mathbb{C}$ is a proper continuous map, and $\{z_n\}$ is any sequence of complex numbers with $\lim_{n\to\infty} |z_n| = \infty$, show that $\lim_{n\to\infty} |f(z_n)| = \infty$.
b) Prove that every proper entire function is a polynomial.

10. Suppose $\{f_k(x)\}$ is a sequence of nonnegative integrable functions on $[0,1]$, and $f_k(x)$ converges to $f(x)$ in measure on $[0,1]$. Suppose, in addition, that

$$\lim_{k\to\infty} \int_{[0,1]} f_k(x)dx = \int_{[0,1]} f(x)dx.$$

Prove that for any measurable subset E of $[0,1]$,

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)dx.$$

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2005, Day 2

This examination will be given in two three-hour sessions, today's being the second part. At each session the examination will have two parts. Answer all three of the questions in Part I (numbered 1–3) and three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate **clearly** which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Second Day—Part I: Answer each of the following three questions

1. Suppose that we are given a continuous function $f_n : \mathbb{R} \to \mathbb{R}$ for each positive integer n, and that the sequence of functions $\{f_n\}$ converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$.

a) Prove that f is a continuous function.

b) Suppose in addition that each f_n is uniformly continuous. Must f be uniformly continuous? Either prove that f is also uniformly continuous, or give an example of a sequence $\{f_n\}$ satisfying all of the hypotheses for which the limit function f is *not* uniformly continuous.

2. Characterize all pairs of groups (G_1, G_2) such that the direct product $G_1 \times G_2$ is cyclic. Show enough work to justify your characterization.

3. In this problem, $U = U_1$ denotes the unit disc $\{z \in \mathbb{C} : |z| < 1\}$, and $U_r = \{z \in \mathbb{C} : |z| < r\}$ denotes the open disc centered at 0 with radius r.

a) Find a holomorphic mapping $f : U \to U$ with no fixed point. That is, there is no $w \in U$ with $f(w) = w$.

b) Suppose $0 < r < 1$, and $g : U \to U_r$ is holomorphic. Prove that g must have exactly one fixed point. **Hint:** Count the roots of $g(z) - z$.

Exam continues on next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Suppose $f \in L^1(\mathbb{R}^1)$. Prove that

$$\sum_{n=-\infty}^{\infty} f(x+n)$$

is absolutely convergent for almost every $x \in \mathbb{R}^1$.

5. Let $f : [0,1] \to \mathbb{R}$ be an increasing function such that

$$\int_0^1 f'(x)\,dx = f(1) - f(0).$$

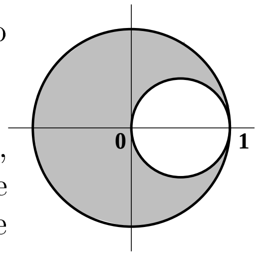
Prove that f is absolutely continuous.

6. Let $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ be a 2×2 matrix with integer entries and determinant 1. Show that A can be written as a product of "elementary" matrices of the form $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$, where m, n are integers.

7. Suppose that R is a commutative ring and that $f(x), g(x)$ are polynomials in $R[x]$ such that $f(x)g(x) = 0$. Show that there is a nonzero element a in R such that $af(x) = 0$.

8. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of closed subsets of a complete metric space (X, d) satisfying $F_{n+1} \subset F_n$ for $n = 1, 2, 3, \ldots$. Assume that the limit of the sequence $\{D_n\}_{n=1}^{\infty}$ of the diameters $D_n = \sup_{x,y \in F_n} d(x, y)$ of F_n is 0. Show that the intersection $\cap_{n=1}^{\infty} F_n$ of F_n is not empty.

9. In this problem, W is the region between two tangent circles as shown. Precisely, $W = \{z \in \mathbb{C} : |z| < 1 \text{ and } |z - \frac{1}{2}| > \frac{1}{2}\}$. Find a conformal mapping of W to the unit disc, $U = \{z \in \mathbb{C} : |z| < 1\}$. Draw pictures of the domain and range of all mappings you may use to describe your answer.



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