RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 27 2009, Day 1

This examination will be given in two three-hour sessions, one today and one tomorrow. At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

First Day—Part I: Answer each of the following three questions

1. Find the following improper integral by using the residue theorem:

$$\int_0^\infty \frac{x^\alpha}{x^2+3x+2}\,dx, \qquad \text{for } 0<\alpha<1$$

Describe clearly the contour of integration and the residue computation.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative Lebesgue measurable function such that $\int_I f(x)dx < \infty$ for all finite intervals $I = [a,b]$. For every Lebesgue-measurable set $A \subset \mathbb{R}$, define

$$\mu(A) = \int_A f(x)dx.$$

(a) Prove that μ is a σ-finite measure on the Lebesgue measurable sets of $\mathbb{R}$.

(b) Prove that if $g : \mathbb{R} \to \mathbb{R}$ is a Lebesgue-measurable function, then

$$\int g(x)d\mu(x) = \int g(x)f(x)dx,$$

in the sense that when either integral exists (in the sense of Lebesgue integration theory), then the other integral also exists, and they are equal.

3. Let A be an abelian group of order 12 (with group operation written additively), and let F be a group homomorphism from A to A. Suppose $F^3 = 0$. Prove that $F^2 = 0$.

First Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $f(z) = 3z^5 - 5z^3 - z - \frac{1}{2}$. How many zeros (counted with multiplicity) does f have in the annulus $\{z \in \mathbb{C},\ 1 < |z| < 2\}$? Prove your statement.

5. Let D be the intersection of the following two domains

$$D_1 := \{z :\ |z-1| < 1\}, \qquad D_2 := \{z :\ |z-i| < 1\}.$$

Exhibit a biholomorphism (a one-to-one and onto holomorphic map whose inverse is holomorphic) from D to the unit disk $\triangle := \{z \in \mathbb{C} :\ |z| < 1\}$ as a composition of suitable rational and elementary transcendental functions. Show transformations of domains made by each of your functions.

6. Let $L^p(\mathbb{R}) = \{f :\ f$ is Lebesgue measurable, $\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p d\mu\right)^{\frac{1}{p}} < \infty\}$.
If $1 < p < q$ and $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, show that $f \in L^r(\mathbb{R})$ for any $p \leq r \leq q$.

7. Show that there are at least 3 nonisomorphic groups of order 75.

8. Let X be a regular polyhedron in 3-space, with V vertices, E edges and F faces. For this problem, you may assume that the group G of isometries of X acts transitively on its vertices, on its edges and on its faces. You may also assume that for each edge, there is an orientation-reversing isometry (a *reflection*) fixing each point on that edge. *Do not use Plato's classification of the regular polyhedra in this problem.*
(1) Show that exactly $2E/V$ edges meet at any vertex.
(2) Show that G has order $4E$.

9. Let V be a nonzero, finite-dimensional vector space over a field F, and assume that the characteristic of F is not 2. Let A, B be endomorphisms of V such that $A^2 = B^3 = (AB)^2 = I$ (the identity), and $Bx \neq x$ for all nonzero vectors $x \in V$. Show that there is a 2-dimensional subspace W of V such that $A(W) \subseteq W$ and $B(W) \subseteq W$.

Day 1 Exam End

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 28, 2009, Day 2

This examination will be given in two three-hour sessions, today's being the second part.
At each session the examination will have two parts.

- Answer all three of the questions in Part I (numbered 1–3)
- Answer three of the six questions in Part II (numbered 4–9). If you work on more than three questions in Part II, indicate (as directed below) which three should be graded. No additional credit will be given for more than three partial solutions. If no three questions are indicated, then the first three questions attempted in the order in which they appear in the examination book(s), and only those, will be the ones graded.

Before handing in your exam:

- Be sure your ID is on each book that you are submitting
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books you are submitting.
- At the top of each book, give a list of the numbers of those problems that appear in the book and that you want to have graded. List them in the order that they appear in the book. The total number of listed problems for all books should be at most 6.
- Within each book make sure that work that you don't want graded is crossed out, or otherwise labeled.

Second Day—Part I: Answer each of the following three questions

1. Let P be a Sylow 5-subgroup of the symmetric group S_{25} on 25 letters.
(1) Determine the order of P.
(2) Show that there are two elements of P that generate P as a group.

2. Let ρ be an outer measure. Prove that A is a ρ-measurable set if and only if for every $\delta > 0$, there is a ρ-measurable set B such that $B \subset A$ and $\rho(A - B) < \delta$. (Here $A - B = A \cap B^c$, the set of elements belonging to A but not B).

3. Let (X, d) be a complete metric space. Assume that X satisfies the following property: For every $\epsilon > 0$ there is finite set $\{x_1, ..., x_n\} \subset X$ such that the union of the n balls of radius ϵ centered at $x_1, \ldots, x_n$ cover X. Prove that X is a compact space.

The exam continues on the next page

Second Day—Part II: Answer three of the following questions. If you work on more than three questions, indicate clearly which three should be graded.

4. Let $f(z) = \sum_{n=1}^{\infty} \dfrac{z^n}{n^{100}}$, For z in the unit disk $\triangle := \{z \in \mathbb{C} : |z| < 1\}$.
(1) Show that $f(z)$ defines a holomorphic function on $\triangle$.
(2) Show that there do not exist an open neighborhood U of 1 and a holomorphic function g on U such that $f = g$ on $U \cap \triangle$.

5. Assume that $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and $\int_{-\infty}^{\infty} |f(x)|dx < \infty$. Show that f vanishes at infinity i.e., $\lim\limits_{|x|\to\infty} f(x) = 0$.

6. Compute a formula for the limit and justify the calculation.

$$\lim_{n\to\infty} \int_0^{\infty} \frac{n^2[\cos(x/n^2) - 1]}{1 + x^3} dx.$$

7. Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix over a field F with characteristic *not* equal to 2, and let $B(x, y) = x^t Ay$ be the associated bilinear form. (Here x and y are column vectors of length n, and x^t is the transposed row vector.) Show that if the field does not have characteristic 2, then there are linearly independent vectors $e_1, \ldots, e_n$ and elements $c_1, \ldots, c_n$ of the field such that $B(\sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j) = \sum_{i=1}^n c_i x_i y_i$.

8. Show that there is no polynomial $z = f(x, y)$ with complex coefficients, and no integer $n > 0$, such that $z^2 = x^n + y^n$.

9. Let $f : \mathbf{R} \to \mathbf{R}$ be a twice differentiable function so that $f(0) = 0$ and $f'(x)$ is decreasing. Prove that $\dfrac{f(x)}{x}$ is decreasing.

Exam Day 2 End