RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2015

Session 1. Algebra

The Qualifying Examination consists of three two-hour sessions. This is the first session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

- Be sure your special exam ID code symbol is on each exam book that you are submitting.
- Label the books at the top as "Book 1 of X", "Book 2 of X", etc., where X is the total number of exam books that you are submitting.
- Within each book make sure that the work that you don't want graded is crossed out or clearly labeled to be ignored.
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Part I. Answer all questions.

1. Let $\mathbb{F}$ be a finite field of order q, with q odd. Show that the following are equivalent:

(a) the equation $x^2 = -1$ has a solution in $\mathbb{F}$

(b) $q \equiv 1 \pmod 4$.

Hint: work with the multiplicative group $\mathbb{F}^\times$ of nonzero elements in $\mathbb{F}$.

2. Recall that the *algebraic multiplicity* of an eigenvalue of a square matrix is defined as its multiplicity as a root of the characteristic polynomial of that matrix. If A is a square matrix with complex entries, let $\exp(A)$ denote the exponential of A, defined as the power series

$$\exp(A) \quad = \quad \sum_{n=0}^{\infty} \frac{1}{n!} A^n \quad = \quad I + A + \frac{1}{2} A^2 + \cdots .$$

Assume all eigenvalues of A are real. If λ is an eigenvalue for A with algebraic multiplicity μ, show that e^λ is an eigenvalue for $\exp(A)$, and has the same algebraic multiplicity μ.

3. Let G be the group $\mathbb{Q}/\mathbb{Z}$, where $\mathbb{Q}$ and $\mathbb{Z}$ are viewed as groups under addition. Prove the following.

(a) Every element of G has finite order.

(b) Every finitely generated subgroup of G is cyclic.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let G be a group of order $2015 = 5 \cdot 13 \cdot 31$.

(a) Prove the existence of normal subgroups of G of orders 13, 31, and 155.
Hint: establish the existence of those subgroups in that order.

(b) Show that G is isomorphic to the direct product of a group of order 13 with a group of order 155.

5. Let $\zeta = \frac{1+\sqrt{-3}}{2}$, and R denote the subring $\mathbb{Z}[\zeta]$ of $\mathbb{C}$.

(a) Show that $R = \mathbb{Z} + \zeta \cdot \mathbb{Z}$.

(b) For $a \in R$, show that $|a|^2 = a\bar{a}$ is an integer, where $\bar{a}$ is the complex conjugate.

(c) For $a \in \mathbb{C}$ show that there are $q \in R$, and $r \in \mathbb{C}$, with

$$a = q + r \text{ and } |r| < 1$$

(d) (Division Algorithm)

Show that for $a, b \in R$ with $b \neq 0$ there are $q, r \in R$ with

$$a = bq + r \text{ and } |r| < |b|$$

(e) Show that R is a principal ideal domain.

End of Session 1

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Session 2. Complex Variables and Advanced Calculus

The Qualifying Examination consists of three two-hour sessions. This is the second session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

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Part I. Answer all questions.

1. Show that if f is an entire function such that

$$f(\mathbb{C}) \cap \{x \in \mathbb{R} : x > 0\} \quad = \quad \emptyset ,$$

then f is a constant.

2. Compute the integral $\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 dx$ using contour integration.

3. For positive real numbers R and r, let

$$E(R,r) \quad = \quad \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \frac{x_1^2 + x_2^2 + x_3^2}{R^2} + \frac{x_4^2}{r^2} \le 1\right\}.$$

Find the volume of $E(R,r)$ by computing an iterated integral.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic functions on an open subset U of the complex plane $\mathbb{C}$. Assume that

1. The sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on compact sets (that is, for every compact subset K of U there exists a real constant C_K such that $|f_n(z)| \le C_K$ for every $z \in K$ and every $n \in \mathbb{N}$).
2. The sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a limit function f (that is, f is a function from U to $\mathbb{C}$, and $\lim_{n\to\infty} f_n(z) = f(z)$ for every $z \in U$).

Prove that f is analytic. (*Hint:* apply the Lebesgue dominated convergence theorem to a suitable contour integral.)

5. Let f be a holomorphic function on the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Assume that $f(0) = \frac{1}{3}$, and that $|f(z)| \le 1$ for all $z \in \mathbb{D}$. Prove that $|f(\frac{1}{20})| \le \frac{2}{5}$.

End of Session 2

RUTGERS UNIVERSITY

GRADUATE PROGRAM IN MATHEMATICS

Written Qualifying Examination

August 2015

Session 3. Real Variables and Elementary Point-Set Topology

The Qualifying Examination consists of three two-hour sessions. This is the third session. The questions for this session are divided into two parts.

Answer **all** of the questions in Part I (numbered 1, 2, 3).

Answer **one** of the questions in Part II (numbered 4, 5).

If you work on both questions in Part II, state **clearly** which one should be graded. No additional credit will be given for more than one of the questions in Part II. If no choice between the two questions is indicated, then the first optional question attempted in the examination book(s) will be the only one graded. **Only material in the examination book(s) will be graded**, and scratch paper will be discarded.

Before handing in your exam at the end of the session:

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Part I. Answer all questions.

1. a) Let m denote Lebesgue measure on $\mathbb{R}$. Prove that the subset A of $L^1(m)$ defined by $A := \{f \in L^1(m) : \int_{\mathbb{R}} |f| dm \leq 1\}$ is closed under pointwise convergence.

b) Prove that the set $B := \{f \in L^1(m) : \int_{\mathbb{R}} |f| dm \geq 1\}$ is not closed under pointwise convergence.

2. a) For α a real number and $\alpha > -1$, prove that $\int_0^\infty x^\alpha e^{-x} dm < \infty$, where m denotes Lebesgue measure on $\mathbb{R}$.

b) For $\alpha > -1$ and k a positive integer, prove that

$$\lim_{k \to \infty} \int_0^k x^\alpha \left(1 - \frac{x}{k}\right)^k dm = \int_0^\infty x^\alpha e^{-x} dm.$$

3. Let (K, d) be a compact metric space which is *well-tied*, which means that for every $\epsilon > 0$, $x \in K$, and $y \in K$, there is a **finite** sequence of points

$$x = x_1,\ x_2, ..., x_n = y \ \text{ in } \ K \qquad \text{such that} \qquad d(x_i, x_{i+1}) \leq \epsilon$$

(n might depend on x and on y).

a) Assume that K can be written as the disjoint union $K = F_1 \cup F_2$, where F_1 and F_2 are both **closed** and nonempty subsets of K. Prove that $d(F_1, F_2) = \inf_{x \in F_1, y \in F_2} d(x, y) > 0$.

b) Show that K is connected.

Hint for part b: Prove that the compact metric space K cannot be written as a disjoint union of two closed, nonempty subsets.

Part II. Answer one of the two questions.
If you work on both questions, indicate clearly which one should be graded.

4. a) Let $[a, b]$ be a closed, bounded interval and $f : [a, b] \to \mathbb{R}$. Give an "epsilon-delta definition" of what it means for f to be "absolutely continuous on $[a, b]$."

b) Assume now that $f : [0,1] \to \mathbb{R}$ has the property that for every ε with $0 < \varepsilon < 1$, the restriction of f to the closed interval $[\epsilon, 1]$ is absolutely continuous. Assume also that there exists some $p > 2$ such that

$$\int_0^1 x|f'(x)|^p \, dm < \infty ,$$

where m denotes Lebesgue measure. Prove that $\lim_{x\to 0^+} f(x)$ exists and is finite.

5. a) For $t \in [0,1]$ and $n \geq 0$, let $u_n(t)$ be the sequence of continuous functions defined by $u_0(t) = 0, \forall t \in [0,1]$, and by the recursion formula

$$u_{n+1}(t) \quad = \quad u_n(t) \; + \; \tfrac{1}{2}(t - u_n(t)^2) \, .$$

Prove that $u_{n+1}(t) \geq u_n(t)$ and $0 \leq u_n(t) \leq \sqrt{t}, \forall t \in [0,1]$ and $\forall n \geq 0$.

b) Prove that the sequence of continuous functions $u_n(t)$ converges uniformly to a continuous function $f(t)$. What is $f(t)$?

End of Session 3