Rutgers - Graduate Program in Mathematics

Written Qualifying Examination

Fall 1995

This exam will be given over two days, in two three hour sessions. Each session will consist of 3 required questions and a choice of 3 out of 6 remaining questions. The basic idea is to ensure that all students at least attempt a range of questions, but one area of weakness should not be overly magnified.

First Day – Part I: Answer each of the following three questions.

1. Let $X = \mathbb{R}$ and let $\tau = \{G \subset X | X \setminus G \text{ is countable}\} \cup \{\emptyset\}$. Show

i) τ is a topology on X;

ii) in the topological space (X, τ) the collection $\mathcal{U}_0$ of neighborhoods of 0 does not have a countable base.

2. Use contour integration to compute

$$\int_0^\infty \frac{dx}{x^4 + 1}.$$

3. Suppose that

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Find a 3×3 orthogonal matrix U such that $U^{-1}AU$ is diagonal.

First Day – Part II: Answer <u>three</u> out of the following six questions.

4. a) How many Sylow 5-subgroups does the alternating group A_5 have?

b) How many Sylow 5-subgroups does the alternating group A_6 have?

c) How many Sylow 5-subgroups does a simple group of order 360 have?
(Hint: use part b).)

5. Let $\{f_n\}$ and $\{g_n\}$ be sequences of measurable functions on $\mathbb{R}$, and let $p > 1$. Assume

i) each f_n is in L_p and $f_n \longrightarrow f$ in L_p;

ii) $g_n \longrightarrow g$ a.e.;

iii) $|g_n| \leq M < \infty$ for each n.

Show that $g_n f_n \longrightarrow gf$ in L_p.

6. Consider the topology on the set of integers $\mathbb{Z}$ for which $\{n + 7^k\mathbb{Z} | k \in \mathbb{Z}\}$ is a basis for the neighborhoods of $n \in \mathbb{Z}$.

i) Show that this is a Hausdorff topology;

ii) Show there is a sequence $\{n_j\}$ in $\mathbb{Z}$ such that $(n_j)^2 \longrightarrow 2$ in this topology.

7. Consider the spiral S in the plane which in polar coordinates has the equation $r = \theta$ for $\theta \geq 0$. Let $G = \mathbb{C} \setminus S$. Show there is a holomorphic function $L : G \longrightarrow \mathbb{C}$ such that $e^{L(z)} = z^2$ for all $z \in G$.

8. Let $\mathbb{Z}_2$ be the field of two elements. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

over $\mathbb{Z}_2$.

a) Show that the characteristic polynomial of A factors as a product of linear polynomials over $\mathbb{Z}_2$.

b) Find a non-singular matrix P over $\mathbb{Z}_2$ so that $P^{-1}AP$ is in Jordan form.

9. Let $\mathbf{v}(x,t)$ be a C^∞ vector field and $\rho(x,t)$ a C^∞ real-valued function defined for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Assume that for any ball $B \subset \mathbb{R}^3$

$$\frac{d}{dt} \iiint\limits_B \rho dx = - \iint\limits_{\partial B} \rho \mathbf{v} \cdot \mathbf{n} dA$$

holds for all t where $\mathbf{n}$ is the exterior unit normal on ∂B and dA is the element of surface area. Show that

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{v}) = 0$$

for all (x,t).

Second Day – Part I: Answer each of the following three questions.

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Assume that $f(x) \longrightarrow L$ as $x \longrightarrow \infty$.

a) Show that if $\lim_{x\to\infty} f'(x)$ exists then $\lim_{x\to\infty} f'(x) = 0$.

b) Is a) true if "$\lim_{x\to\infty} f'(x)$ exists" is replaced by "$f'(x)$ is bounded"?

2. For a ring R let $M_n(R)$ denote the ring of $n \times n$ matrices with entries in r. Prove that if R is a ring with multiplicative identity 1, then every two-sided ideal of $M_n(R)$ is of the form $M_n(I)$, where I is an ideal of R.

3. Let m be the Lebesgue measure on $\mathbb{R}$. Let $A \subset \mathbb{R}$ be a measurable set such that there exists a number b with $0 < b < 1$ and

$$m(A \cap I) < b\, m(I)$$

for all open intervals I. Show that $m(A) = 0$.

Second Day – Part II: Answer <u>three</u> of the following six questions.

4. Let $\{f_n\}$ be a sequence of Lebesgue measurable functions on the interval $(0,1)$ such that

$$\text{i) } \sup \int_0^1 |f_n| dx < \infty$$
$$\text{ii) } f_n \longrightarrow 0 \text{ in measure.}$$

Show that

$$\int_0^1 \sqrt{|f_n|}\, dx \longrightarrow 0.$$

5. Let (X, τ) be a compact metric space. Prove that if $\{U_1, U_2, \ldots, U_k\}$ is an open cover of X, then there exists a closed cover $\{C_1, C_2, \ldots, C_k\}$ with $C_i \subset U_i$ for each i.

6. Let $Q = \{z \in \mathbb{C} | \text{Re } z > 0,\ \text{Im } z > 0\}$ and let $E = \{z \in \mathbb{C} \big| |z| < 1\}$. Give explicitly a holomorphic function $f : Q \longrightarrow E$ which is one-to-one and onto.

7. Let $\mathbb{J}$ denote the set of Gaussian integers; that is,

$$\mathbb{J} = \{\alpha + \beta i | \alpha, \beta \in \mathbb{Z}\}.$$

Let u and v be nonzero Gaussian integers, and let $\bar{\gamma}$ denote the conjugate of the complex number γ.

a) Show that there exist Gaussian integers x, y such that $u\bar{v} = xv\bar{v} + y$ and $y\bar{y} < (v\bar{v})^2$.

b) Show that $\mathbb{J}$ is a Euclidean domain whose degree function is complex absolute value squared.

c) Find the greatest common divisor (in $\mathbb{J}$) of the two Gaussian integers $7 - 6i$ and $7 - 11i$.

8. For what values of λ will the solution $x(t)$ to

$$\begin{aligned} x'(t) &= x^3(t) + \lambda x^2(t) \\ x(0) &= 1 \end{aligned}$$

exist for all $t \geq 0$ and be uniformly bounded. Justify your answer fully.

9. Find two 4×4 matrices A and B such that:

i) A and B have the same characteristic polynomial;

ii) A and B have the same minimal polynomial;

iii) A and B are not similar.